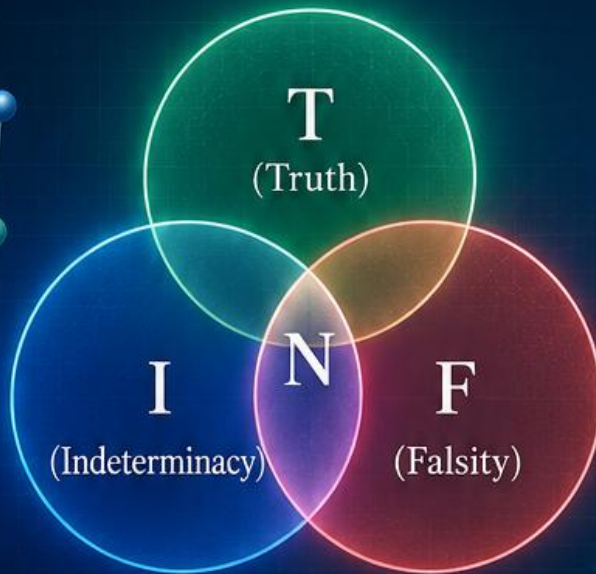
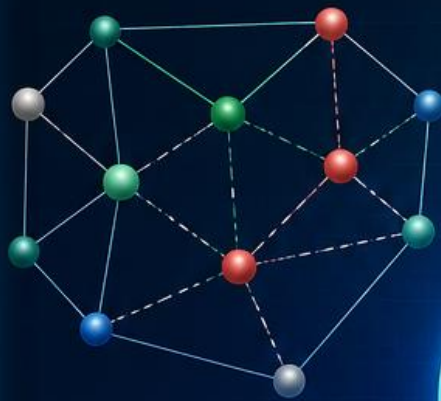


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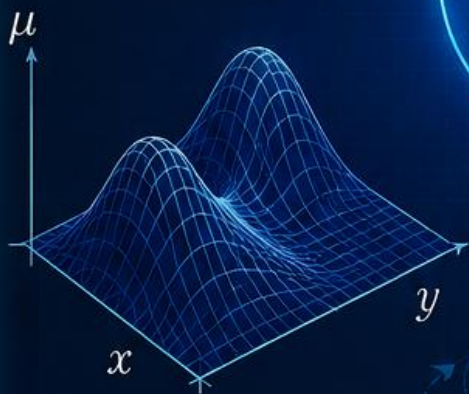
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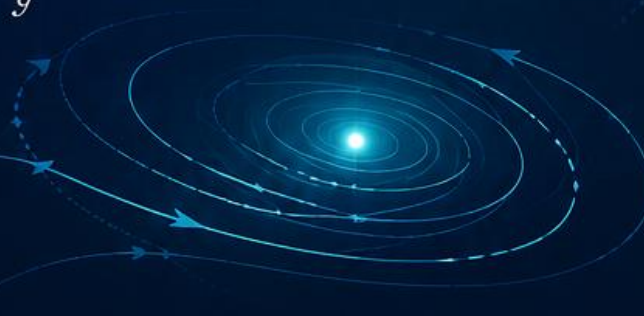
$$[x, y] = -[y, x]$$

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$

$$\phi(L) = \begin{pmatrix} T_1 & I_1 & F_1 \\ T_2 & I_2 & F_2 \\ T_3 & I_3 & F_3 \end{pmatrix}$$



$$M(T_p(x, y), t) \geq 1 - \phi(t, d(x, y)) \text{ for all } t > 0$$



$$x_k \xrightarrow{I, \alpha'} x$$

Mukhtar Ahmad

Florentin Smarandache

Mukhtar Ahmad • Florentin Smarandache

Advanced Topics in Neutrosophic Mathematics

Graphs, Algebra, Analysis, and Applications

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Mukhtar Ahmad • Florentin Smarandache

Advanced Topics in Neutrosophic Mathematics

Graphs, Algebra, Analysis, and Applications



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FOREWORD

The study of uncertainty has undergone remarkable development during the last century. Beginning with classical probability theory and later enriched by the introduction of fuzzy sets, intuitionistic fuzzy sets, and related frameworks, researchers have continually sought mathematical tools capable of representing incomplete, vague, inconsistent, and indeterminate information. As scientific systems become increasingly complex and interconnected, traditional binary models often prove insufficient for describing real-world phenomena characterized by ambiguity and partial knowledge.

A major milestone in this evolution was the introduction of Neutrosophic Theory in the 1990s. By extending the concepts of classical and intuitionistic fuzzy theories, neutrosophy introduced three independent components—truth, indeterminacy, and falsity—allowing uncertainty to be represented in a more flexible and comprehensive manner. Unlike previous frameworks that often constrained these components through complementary relationships, neutrosophic theory permits the independent assessment of each dimension, thereby offering a richer mathematical language for modeling inconsistent, incomplete, and contradictory information.

Over the past decades, neutrosophic mathematics has developed into a broad and active research field encompassing logic, set theory, topology, algebra, graph theory, analysis, optimization, artificial intelligence, decision sciences, engineering, and numerous interdisciplinary applications. The rapid growth of this area has generated new mathematical structures and methodologies that continue to expand both theoretical understanding and practical applicability.

The present volume brings together a collection of recent investigations that reflect several contemporary directions within neutrosophic mathematics. Although the included chapters address different mathematical domains, they share a common objective: the systematic incorporation of uncertainty, indeterminacy, and partial truth into advanced mathematical structures.

The first part of the book focuses on neutrosophic graph theory. The study of parallel and series connections in Min-Max neutrosophic graphs introduces new combinatorial constructions motivated by network modeling and electrical systems. This is complemented by the development of neutrosophic bidirected graphs, which extend graph-theoretical frameworks for representing directional uncertainty and complex network interactions.

The second part explores neutrosophic algebra through the investigation of neutrosophic Lie subalgebras and ideals. By examining images, inverse images, homomorphic properties, and quotient structures, this work contributes to the growing body of research that integrates neutrosophic concepts into abstract algebra and symmetry-related mathematical systems.

The final part of the volume addresses neutrosophic analysis. Fixed-point theory in neutrosophic metric spaces provides powerful tools for studying dynamic processes and optimization problems, while the theory of rough I-statistical convergence in neutrosophic normed spaces extends contemporary approaches to convergence, stability, and robustness in uncertain environments. Together, these contributions demonstrate the breadth of neutrosophic methods and their relevance across diverse branches of modern mathematics.

The motivation for collecting these works into a single volume is twofold. First, despite their distinct mathematical settings, the chapters collectively illustrate the remarkable versatility of neutrosophic theory as a unifying framework. Second, the integration of these studies allows readers to appreciate common themes that transcend disciplinary boundaries, including uncertainty representation, structural generalization, stability analysis, and the development of novel mathematical models. By presenting these investigations together, the volume seeks to provide both a coherent overview of current developments and a foundation for future research.

The future of neutrosophic mathematics appears exceptionally promising. Emerging directions include neutrosophic artificial intelligence, machine learning under uncertainty, complex network analysis, data fusion, decision support systems, quantum information theory, computational algebra, dynamical systems, and interdisciplinary applications in environmental science, medicine, economics, and engineering. As modern scientific challenges increasingly involve incomplete, inconsistent, and rapidly changing information, neutrosophic approaches are expected to play an increasingly significant role in both theoretical investigations and practical problem solving.

It is our hope that this volume will contribute to the continuing advancement of neutrosophic mathematics and inspire further exploration of its rich theoretical foundations and wide-ranging applications. We believe that the ideas presented herein will stimulate new research directions and encourage collaboration among mathematicians, computer scientists, engineers, and researchers from related disciplines.

Mukhtar Ahmad • Florentin Smarandache

Part I — Neutrosophic Graph Theory**COMBINATORIAL STRUCTURE OF PARALLEL AND SERIES CONNECTIONS IN MIN-MAX NEUTROSOPHIC GRAPHS WITH APPLICATIONS TO NETWORK MODELING**MUKHTAR AHMAD¹⁾ AND FLORENTIN SMARANDACHE^{2,*)}¹⁾Algebra Research Group, Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung, Jalan Ganesha no. 10, Bandung, 40132, Indonesia²⁾University of New Mexico, Mathematics, Physics and Natural Sciences Division, Gallup, NM 87301, USA

*corresponding authors email: smarand@unm.edu

ABSTRACT. This paper introduces the concepts of parallel and series connections in Min-max Neutrosophic Graphs and studies their fundamental properties. The proposed operations are described with illustrative examples. In addition, a new class of neutrosophic graphs, called Max-max Neutrosophic Graphs, is defined and analyzed. It is proved that the number of edges in the parallel connection is twice the product of the numbers of edges of the given graphs, whereas for the series connection it is four times the product. The applicability of the proposed model to electrical circuit networks is also discussed.

1. INTRODUCTION

The theory of fuzzy sets, introduced by Zadeh [19], provides a mathematical framework for representing uncertainty, vagueness, and partial membership. This idea has motivated many extensions and applications in discrete mathematics, particularly in graph theory. Rosenfeld [12] initiated the study of fuzzy graphs, which later developed into a rich area involving structural properties, isomorphisms, operations, products, and special classes of fuzzy graphs [2, 4, 5, 14]. In this setting, graph vertices and edges are assigned membership values, allowing graph models to represent networks whose connections are not merely present or absent, but may occur with different degrees of uncertainty.

Further developments were obtained through intuitionistic fuzzy sets introduced by Atanassov [1]. Intuitionistic fuzzy graphs extend fuzzy graphs by considering both membership and non-membership values. This approach has been studied extensively through definitions, operations, complements, regularity, and graph products [9, 10, 11, 3, 6, 7, 8, 13]. These studies show that fuzzy and intuitionistic fuzzy graph models are useful in describing systems where information is incomplete, imprecise, or uncertain.

Another important generalization is provided by neutrosophic set theory, introduced by Smarandache [18, 16, 17]. Neutrosophic sets represent uncertainty by using three independent components, namely truth-membership, indeterminacy-membership, and falsity-membership. This additional indeterminacy component makes neutrosophic models more flexible than classical fuzzy and intuitionistic fuzzy models, especially for systems containing inconsistent or incomplete information. Based on this concept, several types of neutrosophic graphs have been developed, including single-valued neutrosophic graphs,

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Key words and phrases. Neutrosophic graphs, Min-max neutrosophic structure, Parallel connection, Algebraic graph structures, Network topology, Series connection.

mukhkh0786@gmail.com¹; 30125701@mahasiswa.itb.ac.id¹; ORCID: 0009-0006-4597-2278
smarand@unm.edu²; ORCID: 0000-0002-5560-5926.

isolated single-valued neutrosophic graphs, bipolar neutrosophic graphs, and other related graph structures [21, 20, 22, 23, 24]. Recent studies have also considered products, irregularity, hierarchy, and applications of neutrosophic graphs [25, 26, 27].

Graph operations play an essential role in constructing larger and more complex networks from smaller ones. In classical graph theory, operations such as union, product, join, and other forms of composition are commonly used to study structural behavior and network design. Similar ideas have also been investigated in fuzzy, intuitionistic fuzzy, and neutrosophic graph settings. However, many network structures arising in real applications are not only based on general graph operations, but also on connection patterns that resemble circuit constructions. In particular, parallel and series connections are fundamental in electrical circuit networks. These two types of connections motivate the development of new graph operations that can describe how uncertain components interact when arranged in parallel or in series.

In this paper, we introduce the concepts of parallel and series connections in Min-max Neutrosophic Graphs. These operations are defined by combining the neutrosophic information of vertices and edges through minimum and maximum-based rules. Several illustrative examples are provided to clarify the construction and behavior of the proposed operations. We prove that if two Min-max Neutrosophic Graphs are connected in parallel, then the number of edges in the resulting graph is twice the product of the numbers of edges of the given graphs. Meanwhile, for the series connection, the number of edges is four times the product of the numbers of edges of the given graphs.

In addition, we introduce a new class of neutrosophic graphs, called Max-max Neutrosophic Graphs. This class is defined by using maximum-based rules for the neutrosophic membership values associated with graph elements. Fundamental properties of this new class are discussed, and its relation to the proposed connection operations is analyzed. Finally, we discuss the applicability of the proposed model to electrical circuit networks. The proposed framework is expected to provide a mathematical tool for representing circuit systems whose components contain degrees of truth, indeterminacy, and falsity.

2. PRELIMINARIES AND DEFINITIONS

Definition 2.1. A neutrosophic graph on a nonempty set $\widehat{V}$ is defined as $\widehat{G} = (\widehat{T}_\xi, \widehat{I}_\xi, \widehat{F}_\xi, \widehat{T}_\chi, \widehat{I}_\chi, \widehat{F}_\chi)$, where $\widehat{T}_\xi, \widehat{I}_\xi, \widehat{F}_\xi : \widehat{V} \rightarrow [0, 1]$ and $\widehat{T}_\chi, \widehat{I}_\chi, \widehat{F}_\chi : \widehat{V} \times \widehat{V} \rightarrow [0, 1]$ represent the truth-membership, indeterminacy-membership, and falsity-membership functions of vertices and edges, respectively, satisfying

$$\widehat{T}_\chi(b_1, b_2) \leq \min\{\widehat{T}_\xi(b_1), \widehat{T}_\xi(b_2)\},$$

$$\widehat{I}_\chi(b_1, b_2) \leq \max\{\widehat{I}_\xi(b_1), \widehat{I}_\xi(b_2)\},$$

and

$$\widehat{F}_\chi(b_1, b_2) \leq \max\{\widehat{F}_\xi(b_1), \widehat{F}_\xi(b_2)\}$$

for all $b_1, b_2 \in \widehat{V}$.

Definition 2.2. A neutrosophic graph $\widehat{G} = (\widehat{T}_\xi, \widehat{I}_\xi, \widehat{F}_\xi, \widehat{T}_\chi, \widehat{I}_\chi, \widehat{F}_\chi)$ with underlying graph $\widehat{G} = (\widehat{V}, \widehat{E})$ is said to be complete if

$$\widehat{T}_\chi(b_1, b_2) = \min\{\widehat{T}_\xi(b_1), \widehat{T}_\xi(b_2)\},$$

$$\widehat{I}_\chi(b_1, b_2) = \max\{\widehat{I}_\xi(b_1), \widehat{I}_\xi(b_2)\},$$

and

$$\widehat{F}_\chi(b_1, b_2) = \max\{\widehat{F}_\xi(b_1), \widehat{F}_\xi(b_2)\}$$

for all $b_1, b_2 \in \widehat{V}$.

Definition 2.3. A neutrosophic graph $\widehat{G} = (\widehat{T}_\xi, \widehat{I}_\xi, \widehat{F}_\xi, \widehat{T}_\chi, \widehat{I}_\chi, \widehat{F}_\chi)$ with underlying graph $\widehat{G} = (\widehat{V}, \widehat{E})$ is called strong if

$$\widehat{T}_\chi(b_1, b_2) = \min\{\widehat{T}_\xi(b_1), \widehat{T}_\xi(b_2)\},$$

$$\widehat{I}_\chi(b_1, b_2) = \max\{\widehat{I}_\xi(b_1), \widehat{I}_\xi(b_2)\},$$

and

$$\widehat{F}_\chi(b_1, b_2) = \max\{\widehat{F}_\xi(b_1), \widehat{F}_\xi(b_2)\}$$

for all $(b_1, b_2) \in E$.

Definition 2.4. A Min-max Neutrosophic Graph is of the form $\widehat{G} = (\widehat{V}, \widehat{E})$, where

- (a) $\widehat{V} = \{v_1, v_2, \dots, v_n\}$, together with mappings $\widehat{T}_\xi, \widehat{I}_\xi, \widehat{F}_\xi : \widehat{V} \rightarrow [0, 1]$, where $\widehat{T}_\xi(v_i)$, $\widehat{I}_\xi(v_i)$, and $\widehat{F}_\xi(v_i)$ denote the truth-membership, indeterminacy-membership, and falsity-membership values of the vertex v_i , respectively, satisfying

$$0 \leq \widehat{T}_\xi(v_i) + \widehat{I}_\xi(v_i) + \widehat{F}_\xi(v_i) \leq 3 \text{ for every } v_i \in \widehat{V}.$$

- (b) $\widehat{E} \subseteq \widehat{V} \times \widehat{V}$, with mappings $\widehat{T}_\chi, \widehat{I}_\chi, \widehat{F}_\chi : \widehat{V} \times \widehat{V} \rightarrow [0, 1]$ such that

$$\widehat{T}_\chi(v_i, v_j) \leq \min\{\widehat{T}_\xi(v_i), \widehat{T}_\xi(v_j)\},$$

$$\widehat{I}_\chi(v_i, v_j) \leq \max\{\widehat{I}_\xi(v_i), \widehat{I}_\xi(v_j)\},$$

and

$$\widehat{F}_\chi(v_i, v_j) \leq \max\{\widehat{F}_\xi(v_i), \widehat{F}_\xi(v_j)\}$$

for every $(v_i, v_j) \in E$.

Definition 2.5. A Max-min Neutrosophic Graph is of the form $\widehat{G} = (\widehat{V}, \widehat{E})$, where

- (a) $\widehat{V} = \{v_1, v_2, \dots, v_n\}$, together with mappings $\widehat{T}_\xi, \widehat{I}_\xi, \widehat{F}_\xi : \widehat{V} \rightarrow [0, 1]$, satisfying

$$0 \leq \widehat{T}_\xi(v_i) + \widehat{I}_\xi(v_i) + \widehat{F}_\xi(v_i) \leq 3 \text{ for every } v_i \in \widehat{V}.$$

- (b) $\widehat{E} \subseteq \widehat{V} \times \widehat{V}$, with mappings $\widehat{T}_\chi, \widehat{I}_\chi, \widehat{F}_\chi : \widehat{V} \times \widehat{V} \rightarrow [0, 1]$ such that

$$\widehat{T}_\chi(v_i, v_j) \leq \max\{\widehat{T}_\xi(v_i), \widehat{T}_\xi(v_j)\},$$

$$\widehat{I}_\chi(v_i, v_j) \leq \min\{\widehat{I}_\xi(v_i), \widehat{I}_\xi(v_j)\},$$

and

$$\widehat{F}_\chi(v_i, v_j) \leq \min\{\widehat{F}_\xi(v_i), \widehat{F}_\xi(v_j)\}$$

for every $(v_i, v_j) \in E$.

Definition 2.6. A neutrosophic graph $\widehat{G} = (\widehat{V}, \widehat{E})$ is called a complete neutrosophic graph if

$$\widehat{T}_\chi(v_i, v_j) = \min\{\widehat{T}_\xi(v_i), \widehat{T}_\xi(v_j)\},$$

$$\widehat{I}_\chi(v_i, v_j) = \max\{\widehat{I}_\xi(v_i), \widehat{I}_\xi(v_j)\},$$

and

$$\widehat{F}_\chi(v_i, v_j) = \max\{\widehat{F}_\xi(v_i), \widehat{F}_\xi(v_j)\}$$

for all $v_i, v_j \in \widehat{V}$.

3. MAIN RESULTS

4. SERIES AND PARALLEL CONNECTIONS IN NEUTROSOPHIC GRAPHS

Let $\widehat{G}_1 = (\widehat{V}, \widehat{E})$ and $\widehat{G}_2 = (\widehat{V}', \widehat{E}')$ be two neutrosophic graphs, where

$$\widehat{V} = \{v_1, v_2, v_3, \dots, v_n\}$$

and

$$\widehat{V}' = \{v'_1, v'_2, v'_3, \dots, v'_n\},$$

with

$$\widehat{E} \subseteq \widehat{V} \times \widehat{V} \quad \text{and} \quad \widehat{E}' \subseteq \widehat{V}' \times \widehat{V}'.$$

Suppose the corresponding truth-membership, indeterminacy-membership, and falsity-membership functions are given by

$$\widehat{T}_\xi, \widehat{I}_\xi, \widehat{F}_\xi : \widehat{V} \rightarrow [0, 1],$$

$$\widehat{T}_\chi, \widehat{I}_\chi, \widehat{F}_\chi : \widehat{V} \times \widehat{V} \rightarrow [0, 1],$$

and

$$\widehat{T}'_\xi, \widehat{I}'_\xi, \widehat{F}'_\xi : \widehat{V}' \rightarrow [0, 1],$$

$$\widehat{T}'_\chi, \widehat{I}'_\chi, \widehat{F}'_\chi : \widehat{V}' \times \widehat{V}' \rightarrow [0, 1].$$

Definition 4.1. The parallel connection of $\widehat{G}_1$ and $\widehat{G}_2$ with respect to the edges e_1 and e_2 in the neutrosophic graphs, denoted by $P((\widehat{G}_1, e_1) : (\widehat{G}_2, e_2))$, is obtained by deleting the edge $e_1 \in \widehat{G}_1$ and $e_2 \in \widehat{G}_2$, and identifying the end vertices $v_m \in \widehat{G}_1$ of e_1 and $v'_n \in \widehat{G}_2$ of e_2 as a new vertex $\widehat{V}$, such that $\widehat{T}_\xi(v) = \widehat{T}_\xi(v_m) \wedge \widehat{T}_\xi(v'_n)$, $\widehat{I}_\xi(v) = \widehat{I}_\xi(v_m) \vee \widehat{I}_\xi(v'_n)$, and $\widehat{F}_\xi(v) = \widehat{F}_\xi(v_m) \vee \widehat{F}_\xi(v'_n)$.

Similarly, identify the other two end vertices $v_n \in \widehat{G}_1$ of e_1 and $v'_m \in \widehat{G}_2$ of e_2 as a new vertex v' , such that

$$\widehat{T}_\xi(v') = \widehat{T}_\xi(v_n) \wedge \widehat{T}_\xi(v'_m),$$

$$\widehat{I}_\xi(v') = \widehat{I}_\xi(v_n) \vee \widehat{I}_\xi(v'_m),$$

and

$$\widehat{F}_\xi(v') = \widehat{F}_\xi(v_n) \vee \widehat{F}_\xi(v'_m).$$

Finally, introduce a new edge v' , such that

$$\widehat{T}_\chi(v') = \widehat{T}_\chi(e_1) \wedge \widehat{T}_\chi(e_2),$$

$$\widehat{I}_\chi(v') = \widehat{I}_\chi(e_1) \vee \widehat{I}_\chi(e_2),$$

and

$$\widehat{F}_\chi(v') = \widehat{F}_\chi(e_1) \vee \widehat{F}_\chi(e_2).$$

Definition 4.2. The series connection of $\widehat{G}_1$ and $\widehat{G}_2$ with respect to the edges e_1 and e_2 in the neutrosophic graphs, denoted by $S((\widehat{G}_1, e_1) : (\widehat{G}_2, e_2))$, is obtained by deleting the edge $e_1 \in \widehat{G}_1$ and $e_2 \in \widehat{G}_2$, and identifying one of the end vertices $v_m \in \widehat{G}_1$ of e_1 and $v'_n \in \widehat{G}_2$ of e_2 as a new vertex $\widehat{V}$, such that

$$\widehat{T}_\xi(v) = \widehat{T}_\xi(v_m) \wedge \widehat{T}_\xi(v'_n),$$

$$\widehat{I}_\xi(v) = \widehat{I}_\xi(v_m) \vee \widehat{I}_\xi(v'_n),$$

and

$$\widehat{F}_\xi(v) = \widehat{F}_\xi(v_m) \vee \widehat{F}_\xi(v'_n).$$

Finally, introduce a new edge $v_n v'_m$, where $v_n \in \widehat{G}_1$ is an end vertex of e_1 and $v'_m \in \widehat{G}_2$ is an end vertex of e_2 , such that

$$\begin{aligned}\widehat{T}_\chi(v_n v'_m) &= \widehat{T}_\chi(e_1) \wedge \widehat{T}_\chi(e_2), \\ \widehat{I}_\chi(v_n v'_m) &= \widehat{I}_\chi(e_1) \vee \widehat{I}_\chi(e_2),\end{aligned}$$

and

$$\widehat{F}_\chi(v_n v'_m) = \widehat{F}_\chi(e_1) \vee \widehat{F}_\chi(e_2).$$

Example 4.1. Let $\widehat{G}_1 = (\widehat{V}_1, \widehat{E}_1)$ and $\widehat{G}_2 = (\widehat{V}_2, \widehat{E}_2)$ be two neutrosophic graphs. We, take $\widehat{V}_1 = \{a, b, c\}$, $\widehat{V}_2 = \{p, q, r\}$. The neutrosophic vertex values are given as follows:

$$\begin{aligned}a &= (0.7, 0.2, 0.1), & p &= (0.6, 0.1, 0.2), \\ b &= (0.6, 0.3, 0.1), & q &= (0.5, 0.2, 0.2), \\ c &= (0.5, 0.2, 0.2), & r &= (0.4, 0.3, 0.2).\end{aligned}$$

Suppose $e_1 = ab \in \widehat{G}_1$ and $e_2 = pq \in \widehat{G}_2$ with edge values $e_1 = (0.6, 0.3, 0.1)$, $e_2 = (0.5, 0.2, 0.2)$.

Parallel Connection: Identify vertices a and p as a new vertex $\widehat{v}$:

$$\widehat{T}_\xi(v) = 0.7 \wedge 0.6 = 0.6, \quad \widehat{I}_\xi(v) = 0.2 \vee 0.1 = 0.2, \quad \text{and} \quad \widehat{F}_\xi(v) = 0.1 \vee 0.2 = 0.2.$$

Similarly, identify b and q as a new vertex v' :

$$\widehat{T}_\xi(v') = 0.6 \wedge 0.5 = 0.5, \quad \widehat{I}_\xi(v') = 0.3 \vee 0.2 = 0.3, \quad \text{and} \quad \widehat{F}_\xi(v') = 0.1 \vee 0.2 = 0.2.$$

Now, we introduce the new edge $v v'$:

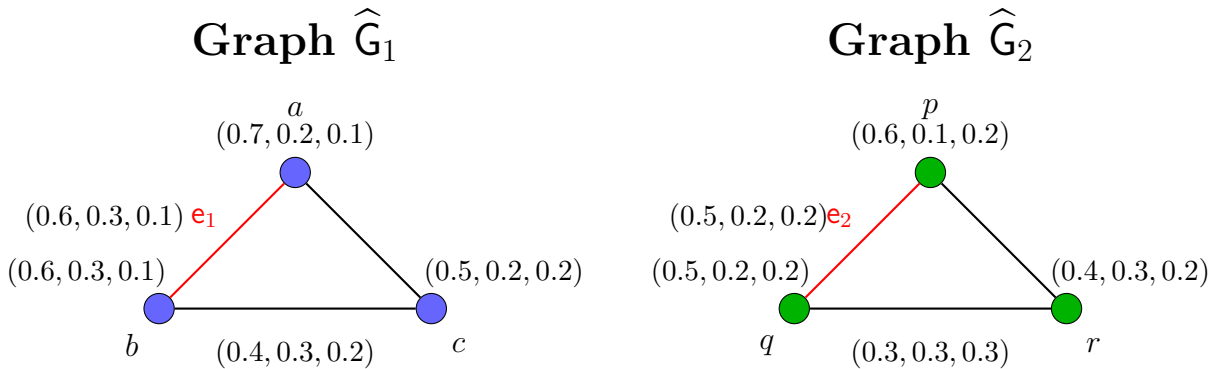
$$\widehat{T}_\chi(v v') = 0.6 \wedge 0.5 = 0.5, \quad \widehat{I}_\chi(v v') = 0.3 \vee 0.2 = 0.3, \quad \text{and} \quad \widehat{F}_\chi(v v') = 0.1 \vee 0.2 = 0.2.$$

Hence, $v v' = (0.5, 0.3, 0.2)$.

The graphical illustration of the parallel connection of the neutrosophic graphs $\widehat{G}_1$ and $\widehat{G}_2$ is shown below. In this construction, the selected edges $e_1 = ab$ and $e_2 = pq$ are removed from the respective graphs, and the corresponding vertices are identified to produce new vertices in the resulting graph. The neutrosophic membership values of the newly formed vertices and edges are determined using the min-max operations on the truth, indeterminacy, and falsity memberships. The resulting graph represents the parallel connection $P((\widehat{G}_1, e_1) : (\widehat{G}_2, e_2))$. For better understanding, the corresponding graphical representation can be seen below.

FIGURE 1 (PARALLEL CONNECTION)

Let $\widehat{G}_1 = (\widehat{V}_1, \widehat{E}_1)$ and $\widehat{G}_2 = (\widehat{V}_2, \widehat{E}_2)$ be two neutrosophic graphs. Let $\widehat{V}_1 = \{a, b, c\}$, $\widehat{V}_2 = \{p, q, r\}$. Consider $e_1 = ab \in \widehat{E}_1$ and $e_2 = pq \in \widehat{E}_2$.



Parallel Connection P

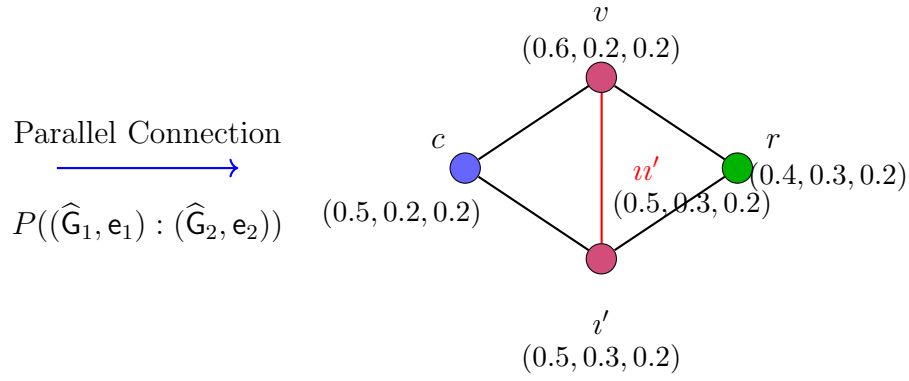
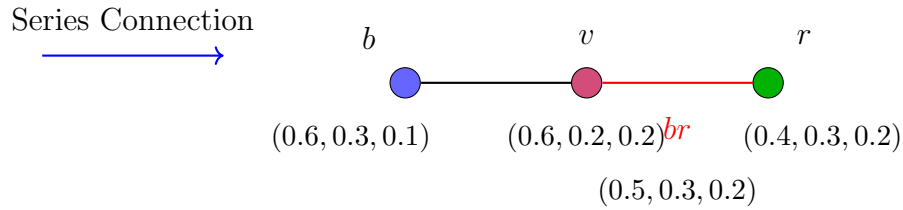
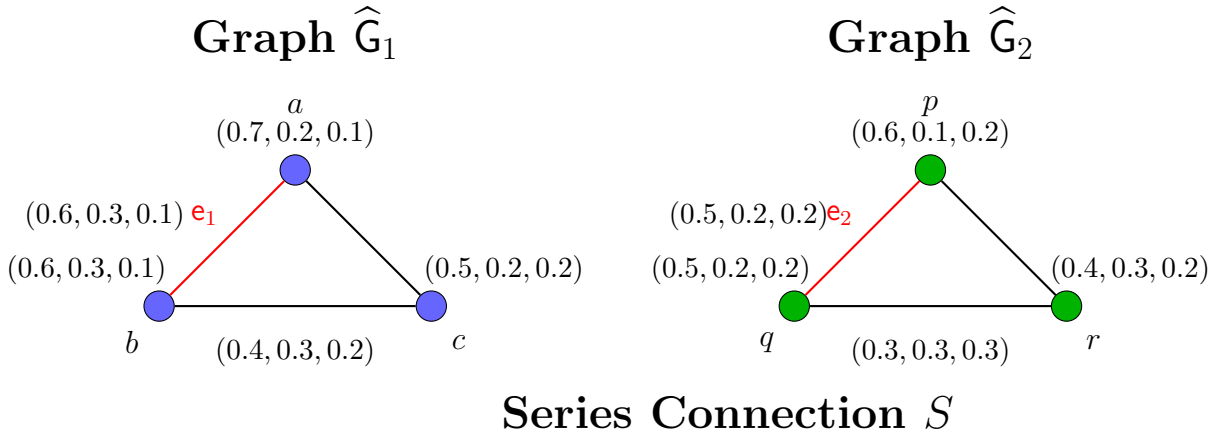


FIGURE 2 (SERIES CONNECTION)



Theorem 4.1. Let $\widehat{G}_1 = (\widehat{V}, \widehat{E})$ and $\widehat{G}_2 = (\widehat{V}', \widehat{E}')$ be two complete Min-max Neutrosophic Graphs. Then, the parallel connection $P(\widehat{G}_1, \widehat{G}_2)$ and the series connection $S(\widehat{G}_1, \widehat{G}_2)$ need not be complete neutrosophic graphs and are not necessarily Min-max Neutrosophic Graphs.

The proof follows from the following example.

Example 4.2. Consider the complete Min-max Neutrosophic cycle graph $\widehat{G}_1 = (\widehat{V}_1, \widehat{E}_1)$ and the complete Min-max Neutrosophic star graph $\widehat{G}_2 = (\widehat{V}_2, \widehat{E}_2)$, shown in the above figures.

For the cycle graph $\widehat{G}_1$, let

$$\widehat{V}_1 = \{a, b, c, d\},$$

with vertex values

$$a = (0.8, 0.1, 0.1), \quad b = (0.7, 0.2, 0.1), \quad c = (0.6, 0.2, 0.2), \quad d = (0.5, 0.3, 0.2).$$

The highlighted edge is

$$e_1 = ab = (0.7, 0.2, 0.1).$$

Similarly, for the star graph $\widehat{G}_2$, let $\widehat{V}_2 = \{p, q, r, s\}$, with

$$p = (0.7, 0.1, 0.2), \quad q = (0.6, 0.2, 0.2), \quad r = (0.5, 0.3, 0.2), \quad s = (0.4, 0.3, 0.3).$$

The selected edge is

$$e_2 = pq = (0.6, 0.2, 0.2).$$

Now, we construct the parallel connection

$$P((\widehat{G}_1, e_1) : (\widehat{G}_2, e_2)),$$

as illustrated in the corresponding figure. After identifying the corresponding vertices, the new connecting edge becomes

$$\omega\omega' = (0.6, 0.2, 0.2).$$

For a complete Min-max Neutrosophic Graph, each edge must satisfy

$$\widehat{T}_\xi(\omega\omega') = \min\{\widehat{T}_\xi(\omega), \widehat{T}_\xi(\omega')\},$$

$$\widehat{I}_\xi(\omega\omega') = \max\{\widehat{I}_\xi(\omega), \widehat{I}_\xi(\omega')\},$$

and

$$\widehat{F}_\xi(\omega\omega') = \max\{\widehat{F}_\xi(\omega), \widehat{F}_\xi(\omega')\}.$$

From the graph,

$$\omega = (0.7, 0.1, 0.2), \quad \omega' = (0.6, 0.2, 0.2).$$

Hence,

$$\widehat{T}_\xi(\omega\omega') = \min\{0.7, 0.6\} = 0.6, \widehat{I}_\xi(\omega\omega') = \max\{0.1, 0.2\} = 0.2, \widehat{F}_\xi(\omega\omega') = \max\{0.2, 0.2\} = 0.2.$$

Although the edge $\omega\omega'$ satisfies the condition, the remaining newly formed edges in the parallel connection do not necessarily preserve the completeness property for all vertex pairs. In particular, some edges are absent between vertices that should be adjacent in a complete graph. Therefore,

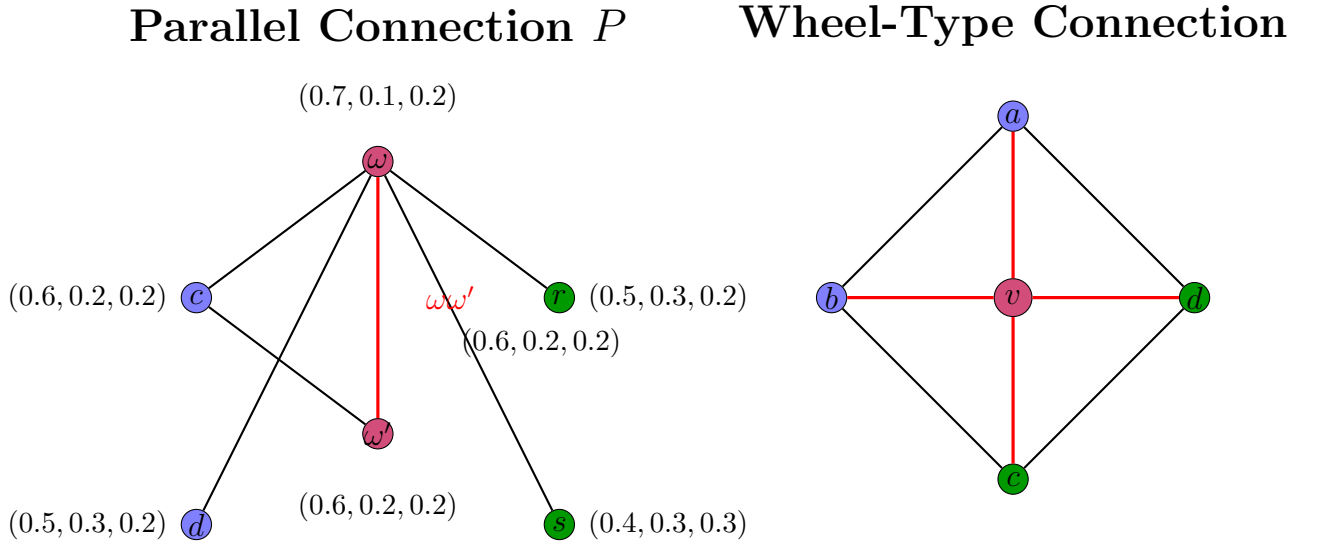
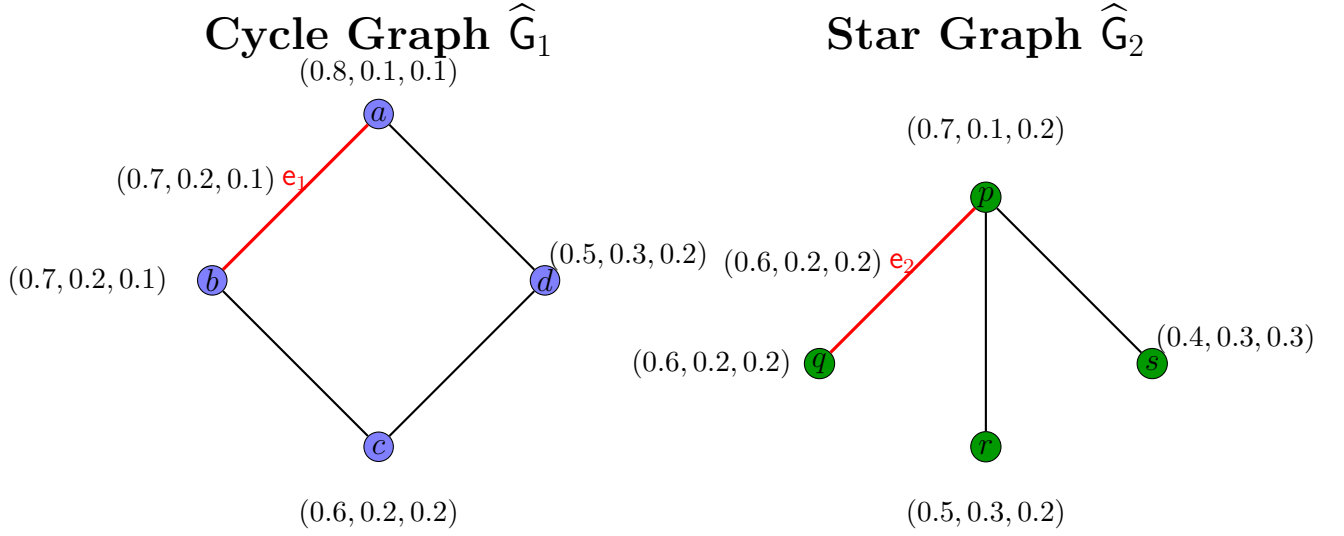
$$P(\widehat{G}_1, \widehat{G}_2)$$

is not necessarily a complete Min-max Neutrosophic Graph.

Similarly, in the wheel-type connection graph, new spokes and connecting edges are introduced. These newly created edges may fail to satisfy the Min-max conditions for truth, indeterminacy, and falsity memberships. Hence, the resulting series-type connection also need not remain complete.

Therefore, $P(\widehat{G}_1, \widehat{G}_2)$ and $S(\widehat{G}_1, \widehat{G}_2)$ are not necessarily complete Min-max Neutrosophic Graphs.

The corresponding graphical representations are given below for better visualization of the proposed neutrosophic graph structures.



Definition 4.3. A Max-max Neutrosophic Graph is of the form $\widehat{G} = (\widehat{V}, \widehat{E})$, where

- (a) $\widehat{V} = \{v_1, v_2, \dots, v_n\}$ together with mappings $\widehat{T}_\xi, \widehat{I}_\xi, \widehat{F}_\xi : \widehat{V} \rightarrow [0, 1]$, where

$$\widehat{T}_\xi(v_i), \widehat{I}_\xi(v_i), \widehat{F}_\xi(v_i)$$

denote the truth-membership, indeterminacy-membership, and falsity-membership values of the vertex $v_i \in \widehat{V}$, respectively, satisfying $0 \leq \widehat{T}_\xi(v_i) + \widehat{I}_\xi(v_i) + \widehat{F}_\xi(v_i) \leq 3$ for every $v_i \in \widehat{V}$, ($i = 1, 2, \dots, n$).

- (b) $\widehat{E} \subseteq \widehat{V} \times \widehat{V}$, with mappings $\widehat{T}_\chi, \widehat{I}_\chi, \widehat{F}_\chi : \widehat{V} \times \widehat{V} \rightarrow [0, 1]$ such that

$$\widehat{T}_\chi(v_i, v_j) \leq \max\{\widehat{T}_\xi(v_i), \widehat{T}_\xi(v_j)\},$$

$$\widehat{I}_\chi(v_i, v_j) \leq \max\{\widehat{I}_\xi(v_i), \widehat{I}_\xi(v_j)\},$$

and

$$\widehat{F}_\chi(v_i, v_j) \leq \max\{\widehat{F}_\xi(v_i), \widehat{F}_\xi(v_j)\}$$

for every $(v_i, v_j) \in E$.

Here, the quadruple $(\iota_i, \widehat{T}_\xi(\iota_i), \widehat{I}_\xi(\iota_i), \widehat{F}_\xi(\iota_i))$ represents the truth-membership, indeterminacy-membership, and falsity-membership values of the vertex $\iota_i \in \widehat{V}$. Similarly, the quadruple

$$((\iota_i, \iota_j), \widehat{T}_\chi(\iota_i, \iota_j), \widehat{I}_\chi(\iota_i, \iota_j), \widehat{F}_\chi(\iota_i, \iota_j))$$

represents the truth-membership, indeterminacy-membership, and falsity-membership values of the edge $(\iota_i, \iota_j) \in E$.

Example 4.3. Consider the graph $\widehat{G} = (\widehat{V}, \widehat{E})$, where $\widehat{V} = \{\iota_1, \iota_2, \iota_3\}$ and $E = \{\iota_1\iota_2, \iota_2\iota_3, \iota_1\iota_3\}$. Now, we define the neutrosophic vertex values as follows:

$$\iota_1 = (0.6, 0.2, 0.1), \quad \iota_2 = (0.7, 0.1, 0.2), \quad \text{and} \quad \iota_3 = (0.5, 0.3, 0.2).$$

The corresponding edge values are given by

$$\iota_1\iota_2 = (0.7, 0.2, 0.2), \quad \iota_2\iota_3 = (0.7, 0.3, 0.2), \quad \text{and} \quad \iota_1\iota_3 = (0.6, 0.3, 0.2).$$

Now, we verify the Max-max neutrosophic conditions. For the edge $\iota_1\iota_2$,

$$\widehat{T}_\chi(\iota_1, \iota_2) = 0.7 \leq \max\{0.6, 0.7\} = 0.7,$$

$$\widehat{I}_\chi(\iota_1, \iota_2) = 0.2 \leq \max\{0.2, 0.1\} = 0.2,$$

and

$$\widehat{F}_\chi(\iota_1, \iota_2) = 0.2 \leq \max\{0.1, 0.2\} = 0.2.$$

Similarly,

$$\widehat{T}_\chi(\iota_2, \iota_3) = 0.7 \leq \max\{0.7, 0.5\} = 0.7,$$

$$\widehat{I}_\chi(\iota_2, \iota_3) = 0.3 \leq \max\{0.1, 0.3\} = 0.3,$$

and

$$\widehat{F}_\chi(\iota_2, \iota_3) = 0.2 \leq \max\{0.2, 0.2\} = 0.2.$$

Also,

$$\widehat{T}_\chi(\iota_1, \iota_3) = 0.6 \leq \max\{0.6, 0.5\} = 0.6,$$

$$\widehat{I}_\chi(\iota_1, \iota_3) = 0.3 \leq \max\{0.2, 0.3\} = 0.3,$$

and

$$\widehat{F}_\chi(\iota_1, \iota_3) = 0.2 \leq \max\{0.1, 0.2\} = 0.2.$$

Hence, all the required conditions are satisfied. Therefore, $\widehat{G} = (\widehat{V}, \widehat{E})$ is a Max-max Neutrosophic Graph.

From the following graph, we observe the structure and neutrosophic labeling of the Max-max Neutrosophic Graph.

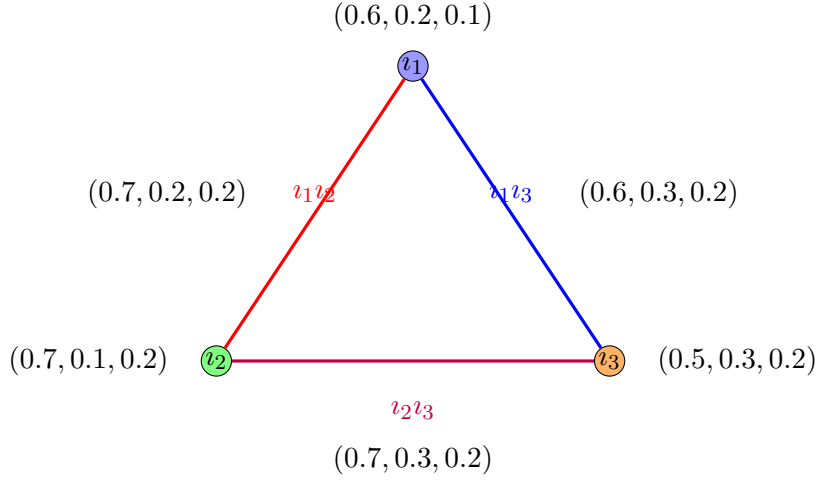


Figure 3. Max-max Neutrosophic Graph

Theorem 4.2. Let $\widehat{G}_1 = (\widehat{V}_1, \widehat{E}_1)$ and $\widehat{G}_2 = (\widehat{V}_2, \widehat{E}_2)$ be two Min-max Neutrosophic Graphs having m and n edges, respectively. Then, the number of possible parallel connections between $\widehat{G}_1$ and $\widehat{G}_2$ is $2mn$.

Proof. Suppose $|\widehat{E}_1| = m$ and $|\widehat{E}_2| = n$. Let $\widehat{E}_1 = \{e_1, e_2, \dots, e_m\}$ and $\widehat{E}_2 = \{e'_1, e'_2, \dots, e'_n\}$.

Each edge of a neutrosophic graph is associated with truth-membership, indeterminacy-membership, and falsity-membership values. Thus, for every edge $e_i = u_1u_2 \in \widehat{E}_1$, there exist neutrosophic edge values $(\widehat{T}_\chi(e_i), \widehat{I}_\chi(e_i), \widehat{F}_\chi(e_i))$, and similarly, for every edge $e'_j = v_1v_2 \in \widehat{E}_2$, there exist $(\widehat{T}_\chi(e'_j), \widehat{I}_\chi(e'_j), \widehat{F}_\chi(e'_j))$. To construct a parallel connection, one edge from $\widehat{G}_1$ and one edge from $\widehat{G}_2$ are selected and deleted.

The number of ways to choose one edge from $\widehat{G}_1$ is m , and the number of ways to choose one edge from $\widehat{G}_2$ is n . Hence, the total number of possible selections of deleted edge pairs is mn .

Now, consider a fixed pair of deleted edges $e_i = u_1u_2$ and $e'_j = v_1v_2$. By the definition of parallel connection in Min-max Neutrosophic Graphs, the end vertices can be identified in two possible ways.

First identification: $u_1 \leftrightarrow v_1, u_2 \leftrightarrow v_2$. The newly formed vertices satisfy

$$\begin{aligned}\widehat{T}_\xi(w_1) &= \widehat{T}_\xi(u_1) \wedge \widehat{T}_\xi(v_1), \\ \widehat{I}_\xi(w_1) &= \widehat{I}_\xi(u_1) \vee \widehat{I}_\xi(v_1), \\ \widehat{F}_\xi(w_1) &= \widehat{F}_\xi(u_1) \vee \widehat{F}_\xi(v_1),\end{aligned}$$

and

$$\begin{aligned}\widehat{T}_\xi(w_2) &= \widehat{T}_\xi(u_2) \wedge \widehat{T}_\xi(v_2), \\ \widehat{I}_\xi(w_2) &= \widehat{I}_\xi(u_2) \vee \widehat{I}_\xi(v_2), \\ \widehat{F}_\xi(w_2) &= \widehat{F}_\xi(u_2) \vee \widehat{F}_\xi(v_2).\end{aligned}$$

The newly introduced edge w_1w_2 has neutrosophic values

$$\begin{aligned}\widehat{T}_\chi(w_1w_2) &= \widehat{T}_\chi(e_i) \wedge \widehat{T}_\chi(e'_j), \\ \widehat{I}_\chi(w_1w_2) &= \widehat{I}_\chi(e_i) \vee \widehat{I}_\chi(e'_j),\end{aligned}$$

and

$$\widehat{F}_\chi(w_1w_2) = \widehat{F}_\chi(e_i) \vee \widehat{F}_\chi(e'_j).$$

Second identification: $u_1 \leftrightarrow v_2$, $u_2 \leftrightarrow v_1$. Again, a valid neutrosophic parallel connection is obtained with corresponding truth-membership, indeterminacy-membership, and falsity-membership values. Therefore, corresponding to each selected pair of deleted edges, exactly two possible neutrosophic parallel connections can be constructed.

Consequently, the total number of possible parallel connections is $2 \times mn = 2mn$. Hence, the number of possible parallel connections between the Min-max Neutrosophic Graphs $\widehat{G}_1$ and $\widehat{G}_2$ is $\boxed{2mn}$. $\square$

Example 4.4. Let $\widehat{G}_1 = (\widehat{V}_1, \widehat{E}_1)$ and $\widehat{G}_2 = (\widehat{V}_2, \widehat{E}_2)$ be two Min-max Neutrosophic Graphs.

Consider $\widehat{V}_1 = \{a, b, c, d\}$ and $\widehat{V}_2 = \{p, q, r, s\}$. The graph $\widehat{G}_1$ is a square graph with edge set $\widehat{E}_1 = \{ab, bc, cd, da\}$, while $\widehat{G}_2$ is a star graph with edge set $\widehat{E}_2 = \{pq, pr, ps\}$. The neutrosophic vertex values of $\widehat{G}_1$ are

$$a = (0.9, 0.1, 0.1), \quad b = (0.8, 0.2, 0.1), \quad c = (0.7, 0.2, 0.2), \quad \text{and} \quad d = (0.6, 0.3, 0.2).$$

The neutrosophic edge values of $\widehat{G}_1$ are

$$ab = (0.8, 0.2, 0.1), \quad bc = (0.7, 0.2, 0.2), \quad cd = (0.6, 0.3, 0.2), \quad \text{and} \quad da = (0.5, 0.3, 0.3).$$

Similarly, the neutrosophic vertex values of $\widehat{G}_2$ are

$$p = (0.8, 0.1, 0.2), \quad q = (0.7, 0.2, 0.2), \quad r = (0.6, 0.2, 0.3), \quad \text{and} \quad s = (0.5, 0.3, 0.3).$$

The neutrosophic edge values of $\widehat{G}_2$ are

$$pq = (0.7, 0.2, 0.2), \quad pr = (0.6, 0.2, 0.3), \quad \text{and} \quad ps = (0.5, 0.3, 0.3).$$

Now, we choose the edges $e_1 = ab \in \widehat{E}_1$ and $e_2 = pq \in \widehat{E}_2$. Delete the edges ab and pq . Identify the vertices $a \leftrightarrow p$ to obtain a new vertex v . Then,

$$\widehat{T}_\xi(v) = \widehat{T}_\xi(a) \wedge \widehat{T}_\xi(p) = \min\{0.9, 0.8\} = 0.8,$$

$$\widehat{I}_\xi(v) = \widehat{I}_\xi(a) \vee \widehat{I}_\xi(p) = \max\{0.1, 0.1\} = 0.1,$$

and

$$\widehat{F}_\xi(v) = \widehat{F}_\xi(a) \vee \widehat{F}_\xi(p) = \max\{0.1, 0.2\} = 0.2.$$

Hence,

$$v = (0.8, 0.1, 0.2).$$

Similarly, $b \leftrightarrow q$ to obtain another new vertex v' . Then,

$$\widehat{T}_\xi(v') = \widehat{T}_\xi(b) \wedge \widehat{T}_\xi(q) = \min\{0.8, 0.7\} = 0.7,$$

$$\widehat{I}_\xi(v') = \widehat{I}_\xi(b) \vee \widehat{I}_\xi(q) = \max\{0.2, 0.2\} = 0.2,$$

and

$$\widehat{F}_\xi(v') = \widehat{F}_\xi(b) \vee \widehat{F}_\xi(q) = \max\{0.1, 0.2\} = 0.2.$$

Thus,

$$v' = (0.7, 0.2, 0.2).$$

Finally, introduce a new edge v' with neutrosophic values

$$\widehat{T}_\chi(v') = \widehat{T}_\chi(ab) \wedge \widehat{T}_\chi(pq) = \min\{0.8, 0.7\} = 0.7,$$

$$\widehat{I}_\chi(v') = \widehat{I}_\chi(ab) \vee \widehat{I}_\chi(pq) = \max\{0.2, 0.2\} = 0.2,$$

and

$$\widehat{F}_\chi(v') = \widehat{F}_\chi(ab) \vee \widehat{F}_\chi(pq) = \max\{0.1, 0.2\} = 0.2.$$

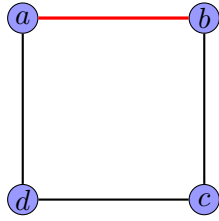
Therefore,

$$u' = (0.7, 0.2, 0.2).$$

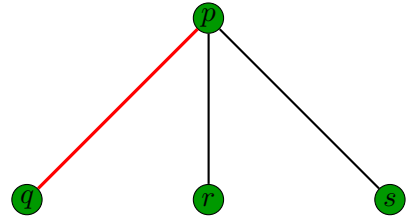
Hence, the parallel connection $P((\widehat{G}_1, e_1) : (\widehat{G}_2, e_2))$ is obtained.

The following figure illustrates the parallel connection of two Min-max Neutrosophic Graphs obtained by identifying the corresponding vertices after deleting the selected edges.

Square Graph $\widehat{G}_1$

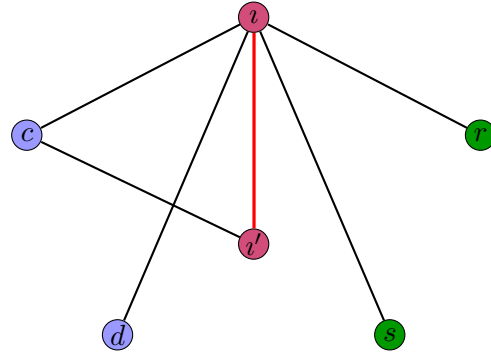


Star Graph $\widehat{G}_2$



Resultant Graph

Parallel Connection



Theorem 4.3. Let $\widehat{G}_1 = (\widehat{V}_1, \widehat{E}_1)$ and $\widehat{G}_2 = (\widehat{V}_2, \widehat{E}_2)$ be two Min-max Neutrosophic Graphs having m and n edges respectively. Then, the number of possible series connections between $\widehat{G}_1$ and $\widehat{G}_2$ is $4mn$.

Proof. Suppose

$$|\widehat{E}_1| = m \quad \text{and} \quad |\widehat{E}_2| = n.$$

Let

$$\widehat{E}_1 = \{e_1, e_2, \dots, e_m\}, \quad \widehat{E}_2 = \{e'_1, e'_2, \dots, e'_n\}.$$

To construct a series connection, one edge from each graph is selected and deleted. Hence, the number of ways of choosing one edge from $\widehat{G}_1$ and one edge from $\widehat{G}_2$ is mn . Let $e_i = u_1u_2 \in \widehat{E}_1$, $e'_j = v_1v_2 \in \widehat{E}_2$ be a chosen pair of edges. In a series connection, one vertex from each deleted edge is identified to form a single vertex. There are: $2 \times 2 = 4$ possible ways to choose one endpoint from e_i and one endpoint from e'_j :

Case 1:

$$u_1 \leftrightarrow v_1$$

Case 2:

$$u_1 \leftrightarrow v_2$$

Case 3:

$$u_2 \leftrightarrow v_1$$

Case 4:

$$u_2 \leftrightarrow v_2$$

Each identification produces a valid neutrosophic series connection, since the resulting vertex and edge memberships are defined using the Min-max neutrosophic operations:

$$\widehat{T}_\xi = \min, \quad \widehat{I}_\xi = \max, \quad \widehat{F}_\xi = \max.$$

Thus, for each pair of edges, there are 4 possible series constructions.

Therefore, total number of series connections is $4 \times mn = 4mn$. Hence proved. $\square$

EXAMPLE OF SERIES CONNECTION IN GRAPHS

Let

$$\widehat{G}_1 = (\widehat{V}_1, \widehat{E}_1), \quad \widehat{V}_1 = \{a, b, c\}, \quad \widehat{E}_1 = \{ab, bc\}$$

and

$$\widehat{G}_2 = (\widehat{V}_2, \widehat{E}_2), \quad \widehat{V}_2 = \{b_1, b_2, z\}, \quad \widehat{E}_2 = \{b_1b_2, b_2z\}.$$

Both graphs are path graphs.

The number of edges are:

$$|\widehat{E}_1| = 2, \quad |\widehat{E}_2| = 2.$$

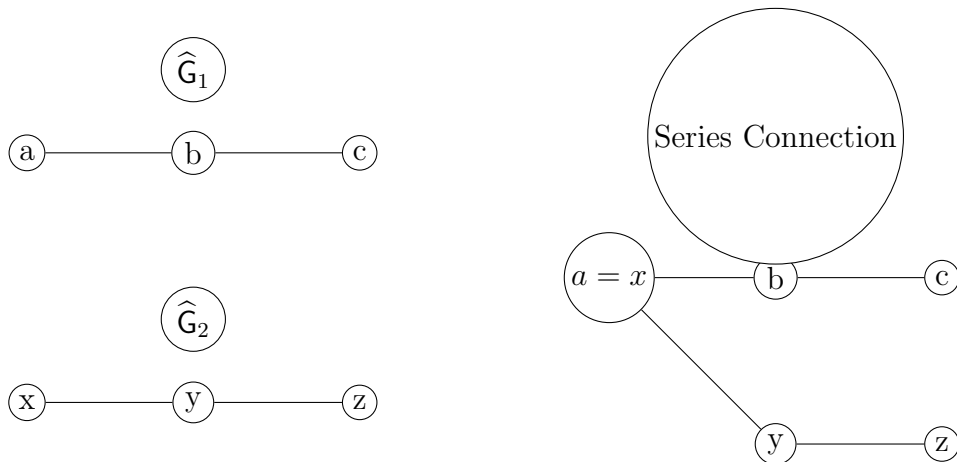
Thus, the number of possible series connections is:

$$4mn = 4 \times 2 \times 2 = 16.$$

A series connection is formed by:

- (1) selecting one edge from each graph,
- (2) deleting those edges,
- (3) identifying one endpoint from each edge.

Example: Identify $a \equiv b_1$.



5. CONCLUSION

In this paper, we introduced and studied the concepts of parallel and series connections in Min-Max Neutrosophic Graphs. These operations extend classical graph-theoretic constructions into the neutrosophic environment, where uncertainty is modeled using truth, indeterminacy, and falsity membership functions. The structural behavior of these connections was analyzed, and exact combinatorial expressions were derived. It was shown that the number of possible parallel connections is $2mn$, while the number of series connections is $4mn$, where m and n are the number of edges in the respective graphs.

Furthermore, a new class of neutrosophic graphs, called Max-Max Neutrosophic Graphs, was introduced to broaden the framework of uncertainty-based graph modeling. The obtained results generalize existing work in fuzzy and intuitionistic fuzzy graphs and provide a unified combinatorial interpretation of graph connection operations under neutrosophic settings.

The proposed structures have potential applications in network theory, decision-making systems, and electrical circuit modeling, where uncertainty and indeterminacy play a significant role. Future work may focus on extending these operations to weighted neutrosophic graphs, hypergraphs, and dynamic network systems, as well as exploring algebraic and spectral properties of such constructions.

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ON NEUTROSOPHIC BIDIRECTED GRAPHS AND THEIR STRUCTURAL PROPERTIES

MUKHTAR AHMAD¹⁾ AND FLORENTIN SMARANDACHE^{2,*)}

¹⁾Algebra Research Group, Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung, Jalan Ganesha no. 10, Bandung, 40132, Indonesia

²⁾University of New Mexico, Mathematics, Physics and Natural Sciences Division, Gallup, NM 87301, USA

*corresponding authors email: smarand@unm.edu

ABSTRACT. In this paper, we introduce the concept of neutrosophic bidirected graphs as a generalization of intuitionistic fuzzy bidirected graphs by incorporating truth-membership, indeterminacy-membership, and falsity-membership functions on vertices and edges together with a bidirection mapping. Several fundamental properties and structural results are established. In particular, we investigate reductions to fuzzy and crisp bidirected graphs, characterize neutrosophic directed graphs induced by bidirections, and study operations such as intersection and union. Furthermore, strong neutrosophic bidirected graphs, homomorphisms, and isomorphisms are discussed. The proposed structure provides a useful mathematical model for handling uncertainty and indeterminacy in bidirected network structures.

1. INTRODUCTION

Graph theory has become an essential branch of mathematics with wide-ranging applications in computer science, communication systems, artificial intelligence, transportation networks, optimization, and decision sciences. Classical graph models, however, are often inadequate for representing uncertainty, inconsistency, and incomplete information that naturally arise in practical systems. To overcome these difficulties, several generalized graph models based on fuzzy and neutrosophic frameworks have been developed.

The theory of fuzzy sets, introduced by Zadeh [1], provides an effective mathematical tool for handling uncertainty through membership functions. Rosenfeld [2] initiated the concept of fuzzy graphs by extending fuzzy set theory to graph structures. Later, Atanassov [3] generalized fuzzy sets to intuitionistic fuzzy sets by introducing both membership and non-membership degrees. Since then, intuitionistic fuzzy graphs and their algebraic and structural properties have been studied extensively by many researchers [4, 5, 6].

In recent years, neutrosophic set theory introduced by Smarandache [7] has attracted considerable attention because of its ability to characterize uncertainty, indeterminacy, and inconsistency simultaneously through truth-membership, indeterminacy-membership, and falsity-membership functions. Compared with intuitionistic fuzzy theory, neutrosophic theory offers a more flexible framework for modeling incomplete and inconsistent information. The development of neutrosophic graph theory has therefore become an active area of research. Several important contributions related to neutrosophic graphs, neutrosophic digraphs, and their applications can be found in [8, 9, 10, 11, 12].

Bidirected graphs constitute another important generalization of graph structures in which each edge possesses local orientations at its end vertices. These graphs are useful in network flow analysis, communication systems, transportation models, electrical circuit analysis, and

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mukhkh0786@gmail.com¹; 30125701@mahasiswa.itb.ac.id¹; ORCID: 0009-0006-4597-2278
smarand@unm.edu²; ORCID: 0000-0002-5560-5926.

combinatorial optimization. Fuzzy and intuitionistic fuzzy bidirected graph models were introduced to incorporate uncertainty into bidirectional network structures. However, those models cannot explicitly represent indeterminate information.

Motivated by the above observations, in this paper we introduce the concept of neutrosophic bidirected graphs as a natural extension of intuitionistic fuzzy bidirected graphs. The proposed framework combines neutrosophic information with bidirectional graph structures by assigning truth-membership, indeterminacy-membership, and falsity-membership values to vertices and edges together with a bidirection mapping. Several fundamental properties and structural characterizations are established. In particular, we investigate reductions to fuzzy and crisp bidirected graphs, induced neutrosophic directed graphs, intersection and union operations, strong neutrosophic bidirected graphs, homomorphisms, and isomorphisms.

The obtained results provide a generalized mathematical setting for modeling uncertainty and indeterminacy in bidirected network systems and may serve as a foundation for further studies in neutrosophic graph theory and its applications in intelligent systems, data analysis, optimization, and decision-making problems.

2. FUNDAMENTAL CONCEPTS

Definition 2.1 ([13]). *A neutrosophic set $\tilde{A}$ on $\tilde{X}$ is characterized by three functions*

$$\top_{\tilde{A}} : \tilde{X} \rightarrow [0, 1], \quad \text{I}_{\tilde{A}} : \tilde{X} \rightarrow [0, 1], \quad \text{and} \quad \text{F}_{\tilde{A}} : \tilde{X} \rightarrow [0, 1],$$

called the truth-membership, indeterminacy-membership, and falsity-membership functions, respectively, such that

$$0 \leq \top_{\tilde{A}}(\partial) + \text{I}_{\tilde{A}}(\partial) + \text{F}_{\tilde{A}}(\partial) \leq 3$$

for all $\partial \in \tilde{X}$.

Definition 2.2. *Let $\mathcal{G}^* = (\mathcal{V}, \text{E})$ be a crisp undirected simple graph. A neutrosophic graph (NG) on $\mathcal{G}^*$ is defined as*

$$\mathcal{G} = ((\top, \text{I}, \text{F}), (\top_{\text{E}}, \text{I}_{\text{E}}, \text{F}_{\text{E}})),$$

where

$$\top, \text{I}, \text{F} : \mathcal{V} \rightarrow [0, 1],$$

and

$$\top_{\text{E}}, \text{I}_{\text{E}}, \text{F}_{\text{E}} : \text{E} \rightarrow [0, 1],$$

are the truth-membership, indeterminacy-membership, and falsity-membership functions on vertices and edges, respectively, satisfying

$$0 \leq \top(\partial) + \text{I}(\partial) + \text{F}(\partial) \leq 3,$$

$$0 \leq \top_{\text{E}}(\{\zeta, \partial\}) + \text{I}_{\text{E}}(\{\zeta, \partial\}) + \text{F}_{\text{E}}(\{\zeta, \partial\}) \leq 3,$$

and

$$\top_{\text{E}}(\{\zeta, \partial\}) \leq \min\{\top(\zeta), \top(\partial)\},$$

$$\text{I}_{\text{E}}(\{\zeta, \partial\}) \leq \max\{\text{I}(\zeta), \text{I}(\partial)\},$$

$$\text{F}_{\text{E}}(\{\zeta, \partial\}) \geq \max\{\text{F}(\zeta), \text{F}(\partial)\},$$

for all $\{\zeta, \partial\} \in \text{E}$.

Example 2.1. *Consider the crisp undirected simple graph $\mathcal{G}^* = (\mathcal{V}, \text{E})$, where $\mathcal{V} = \{\partial_1, \partial_2, \partial_3, \partial_4\}$ and $\text{E} = \{\{\partial_1, \partial_2\}, \{\partial_1, \partial_3\}, \{\partial_2, \partial_3\}, \{\partial_3, \partial_4\}\}$. Define the neutrosophic vertex functions by*

$$\top(\partial_1) = 0.9, \quad \text{I}(\partial_1) = 0.1, \quad \text{F}(\partial_1) = 0.1,$$

$$\top(\partial_2) = 0.7, \quad \text{I}(\partial_2) = 0.3, \quad \text{F}(\partial_2) = 0.2,$$

$$\top(\partial_3) = 0.6, \quad \text{I}(\partial_3) = 0.4, \quad \text{F}(\partial_3) = 0.3,$$

$$\top(\partial_4) = 0.5, \quad \text{I}(\partial_4) = 0.5, \quad \text{F}(\partial_4) = 0.4.$$

Clearly,

$$0 \leq \top(\partial_i) + \mathbf{l}(\partial_i) + \mathbf{F}(\partial_i) \leq 3$$

for each $i = 1, 2, 3, 4$, since

$$0.9 + 0.1 + 0.1 = 1.1,$$

$$0.7 + 0.3 + 0.2 = 1.2,$$

$$0.6 + 0.4 + 0.3 = 1.3,$$

and

$$0.5 + 0.5 + 0.4 = 1.4.$$

Now, we define the neutrosophic edge functions as follows:

$$\top_{\mathbf{E}}(\{\partial_1, \partial_2\}) = 0.7, \quad \mathbf{l}_{\mathbf{E}}(\{\partial_1, \partial_2\}) = 0.3, \quad \mathbf{F}_{\mathbf{E}}(\{\partial_1, \partial_2\}) = 0.2,$$

$$\top_{\mathbf{E}}(\{\partial_1, \partial_3\}) = 0.6, \quad \mathbf{l}_{\mathbf{E}}(\{\partial_1, \partial_3\}) = 0.4, \quad \mathbf{F}_{\mathbf{E}}(\{\partial_1, \partial_3\}) = 0.3,$$

$$\top_{\mathbf{E}}(\{\partial_2, \partial_3\}) = 0.5, \quad \mathbf{l}_{\mathbf{E}}(\{\partial_2, \partial_3\}) = 0.4, \quad \mathbf{F}_{\mathbf{E}}(\{\partial_2, \partial_3\}) = 0.3,$$

$$\top_{\mathbf{E}}(\{\partial_3, \partial_4\}) = 0.5, \quad \mathbf{l}_{\mathbf{E}}(\{\partial_3, \partial_4\}) = 0.5, \quad \mathbf{F}_{\mathbf{E}}(\{\partial_3, \partial_4\}) = 0.4.$$

We verify the defining conditions.

For the edge $\{\partial_1, \partial_2\}$,

$$\top_{\mathbf{E}}(\{\partial_1, \partial_2\}) = 0.7 \leq \min\{0.9, 0.7\} = 0.7,$$

$$\mathbf{l}_{\mathbf{E}}(\{\partial_1, \partial_2\}) = 0.3 \leq \max\{0.1, 0.3\} = 0.3,$$

and

$$\mathbf{F}_{\mathbf{E}}(\{\partial_1, \partial_2\}) = 0.2 \geq \max\{0.1, 0.2\} = 0.2.$$

For the edge $\{\partial_1, \partial_3\}$,

$$0.6 \leq \min\{0.9, 0.6\} = 0.6,$$

$$0.4 \leq \max\{0.1, 0.4\} = 0.4,$$

and

$$0.3 \geq \max\{0.1, 0.3\} = 0.3.$$

For the edge $\{\partial_2, \partial_3\}$,

$$0.5 \leq \min\{0.7, 0.6\} = 0.6,$$

$$0.4 \leq \max\{0.3, 0.4\} = 0.4,$$

and

$$0.3 \geq \max\{0.2, 0.3\} = 0.3.$$

For the edge $\{\partial_3, \partial_4\}$,

$$0.5 \leq \min\{0.6, 0.5\} = 0.5,$$

$$0.5 \leq \max\{0.4, 0.5\} = 0.5,$$

and

$$0.4 \geq \max\{0.3, 0.4\} = 0.4.$$

Moreover,

$$0 \leq \top_{\mathbf{E}}(\hat{\mathbf{g}}) + \mathbf{l}_{\mathbf{E}}(\hat{\mathbf{g}}) + \mathbf{F}_{\mathbf{E}}(\hat{\mathbf{g}}) \leq 3$$

for every edge $\hat{\mathbf{g}} \in \mathbf{E}$. Hence, $\mathcal{G} = ((\top, \mathbf{l}, \mathbf{F}), (\top_{\mathbf{E}}, \mathbf{l}_{\mathbf{E}}, \mathbf{F}_{\mathbf{E}}))$ is a neutrosophic graph on $\mathcal{G}^*$.

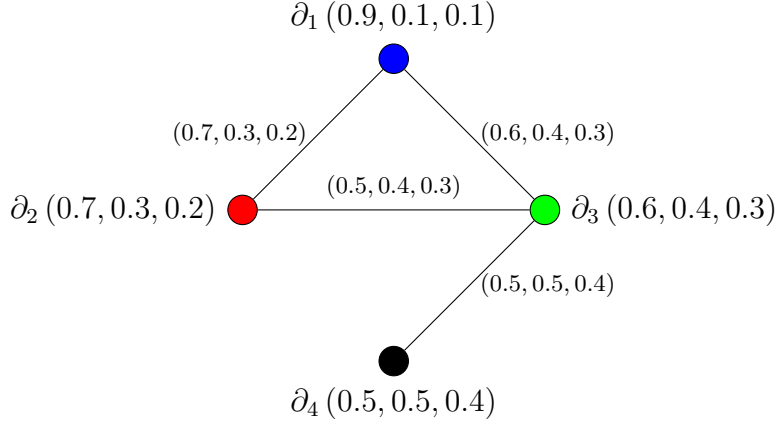


FIGURE 1. Neutrosophic graph corresponding to Example 2.1

Definition 2.3. Let $\vec{\mathcal{G}} = (\mathcal{V}, \tilde{\mathcal{A}})$ be a crisp directed simple graph. A neutrosophic directed graph (NDG) on $\vec{\mathcal{G}}$ is defined as $\vec{\mathcal{G}}_N = ((\top, \mathbf{l}, \mathbf{F}), (\top_{\tilde{\mathcal{A}}}, \mathbf{l}_{\tilde{\mathcal{A}}}, \mathbf{F}_{\tilde{\mathcal{A}}}))$, where

$$\top, \mathbf{l}, \mathbf{F} : \mathcal{V} \rightarrow [0, 1], \text{ and } \top_{\tilde{\mathcal{A}}}, \mathbf{l}_{\tilde{\mathcal{A}}}, \mathbf{F}_{\tilde{\mathcal{A}}} : \tilde{\mathcal{A}} \rightarrow [0, 1],$$

are the truth-membership, indeterminacy-membership, and falsity-membership functions on vertices and arcs, respectively, satisfying

$$0 \leq \top(\partial) + \mathbf{l}(\partial) + \mathbf{F}(\partial) \leq 3,$$

$$0 \leq \top_{\tilde{\mathcal{A}}}(\iota_1, \iota_2) + \mathbf{l}_{\tilde{\mathcal{A}}}(\iota_1, \iota_2) + \mathbf{F}_{\tilde{\mathcal{A}}}(\iota_1, \iota_2) \leq 3,$$

and

$$\top_{\tilde{\mathcal{A}}}(\iota_1, \iota_2) \leq \min\{\top(\iota_1), \top(\iota_2)\},$$

$$\mathbf{l}_{\tilde{\mathcal{A}}}(\iota_1, \iota_2) \leq \max\{\mathbf{l}(\iota_1), \mathbf{l}(\iota_2)\},$$

$$\mathbf{F}_{\tilde{\mathcal{A}}}(\iota_1, \iota_2) \geq \max\{\mathbf{F}(\iota_1), \mathbf{F}(\iota_2)\},$$

for all $(\iota_1, \iota_2) \in \tilde{\mathcal{A}}$.

Example 2.2. Consider the crisp directed graph $\vec{\mathcal{G}} = (\mathcal{V}, \tilde{\mathcal{A}})$, where $\mathcal{V} = \{\partial_1, \partial_2, \partial_3\}$ and $\tilde{\mathcal{A}} = \{(\partial_1, \partial_2), (\partial_2, \partial_3), (\partial_1, \partial_3)\}$. Define the neutrosophic vertex functions

$$\top(\partial_1) = 0.8, \quad \mathbf{l}(\partial_1) = 0.2, \quad \mathbf{F}(\partial_1) = 0.1,$$

$$\top(\partial_2) = 0.6, \quad \mathbf{l}(\partial_2) = 0.3, \quad \mathbf{F}(\partial_2) = 0.2,$$

$$\top(\partial_3) = 0.7, \quad \mathbf{l}(\partial_3) = 0.4, \quad \mathbf{F}(\partial_3) = 0.2.$$

Clearly,

$$0 \leq \top(\partial_i) + \mathbf{l}(\partial_i) + \mathbf{F}(\partial_i) \leq 3$$

for each $i = 1, 2, 3$, since

$$0.8 + 0.2 + 0.1 = 1.1,$$

$$0.6 + 0.3 + 0.2 = 1.1,$$

and

$$0.7 + 0.4 + 0.2 = 1.3.$$

Now define the neutrosophic arc functions as follows:

$$\top_{\tilde{\mathcal{A}}}(\partial_1, \partial_2) = 0.5, \quad \mathbf{l}_{\tilde{\mathcal{A}}}(\partial_1, \partial_2) = 0.3, \quad \mathbf{F}_{\tilde{\mathcal{A}}}(\partial_1, \partial_2) = 0.2,$$

$$\top_{\tilde{\mathcal{A}}}(\partial_2, \partial_3) = 0.6, \quad \mathbf{l}_{\tilde{\mathcal{A}}}(\partial_2, \partial_3) = 0.4, \quad \mathbf{F}_{\tilde{\mathcal{A}}}(\partial_2, \partial_3) = 0.2,$$

$$\top_{\tilde{\mathcal{A}}}(\partial_1, \partial_3) = 0.7, \quad \mathbf{l}_{\tilde{\mathcal{A}}}(\partial_1, \partial_3) = 0.4, \quad \mathbf{F}_{\tilde{\mathcal{A}}}(\partial_1, \partial_3) = 0.2.$$

We verify the required conditions.

For the arc (∂_1, ∂_2) ,

$$\top_{\tilde{A}}(\partial_1, \partial_2) = 0.5 \leq \min\{0.8, 0.6\} = 0.6,$$

$$\mathbf{l}_{\tilde{A}}(\partial_1, \partial_2) = 0.3 \leq \max\{0.2, 0.3\} = 0.3,$$

and

$$\mathbf{F}_{\tilde{A}}(\partial_1, \partial_2) = 0.2 \geq \max\{0.1, 0.2\} = 0.2.$$

Similarly, for the arc (∂_2, ∂_3) ,

$$0.6 \leq \min\{0.6, 0.7\} = 0.6,$$

$$0.4 \leq \max\{0.3, 0.4\} = 0.4,$$

and

$$0.2 \geq \max\{0.2, 0.2\} = 0.2.$$

For the arc (∂_1, ∂_3) ,

$$0.7 \leq \min\{0.8, 0.7\} = 0.7,$$

$$0.4 \leq \max\{0.2, 0.4\} = 0.4,$$

and

$$0.2 \geq \max\{0.1, 0.2\} = 0.2.$$

Moreover,

$$0 \leq \top_{\tilde{A}}(\iota_1, \iota_2) + \mathbf{l}_{\tilde{A}}(\iota_1, \iota_2) + \mathbf{F}_{\tilde{A}}(\iota_1, \iota_2) \leq 3$$

for every arc $(\iota_1, \iota_2) \in \tilde{\mathbf{A}}$. Hence, $\vec{\mathcal{G}}_N = ((\top, \mathbf{l}, \mathbf{F}), (\top_{\tilde{A}}, \mathbf{l}_{\tilde{A}}, \mathbf{F}_{\tilde{A}}))$ is a neutrosophic directed graph.

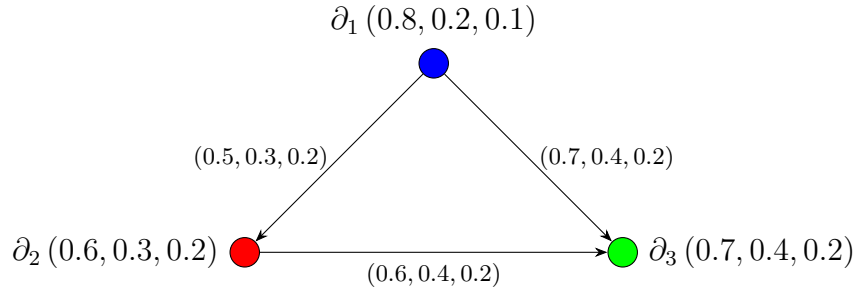


FIGURE 2. Neutrosophic directed graph corresponding to Example 2.2

3. MAIN RESULTS

Definition 3.1. Let $\mathcal{G}^* = (\mathcal{V}, \mathbf{E})$ be a crisp undirected simple graph, where $\mathcal{V}$ is a nonempty set of vertices and $\mathbf{E} \subseteq \{\{\zeta, \partial\} : \zeta, \partial \in \mathcal{V}, \zeta \neq \partial\}$ is its edge set.

A neutrosophic bidirected graph is a septuple $\mathcal{B}_N = (\top, \mathbf{l}, \mathbf{F}, \top_{\mathbf{E}}, \mathbf{l}_{\mathbf{E}}, \mathbf{F}_{\mathbf{E}}, \xi)$, where

$$\top, \mathbf{l}, \mathbf{F} : \mathcal{V} \rightarrow [0, 1], \quad \top_{\mathbf{E}}, \mathbf{l}_{\mathbf{E}}, \mathbf{F}_{\mathbf{E}} : \mathbf{E} \rightarrow [0, 1], \quad \text{and } \xi : \mathcal{V} \times \mathbf{E} \rightarrow \{-1, 0, 1\},$$

satisfying the following conditions:

(1) For every $\partial \in \mathcal{V}$,

$$0 \leq \top(\partial) + \mathbf{l}(\partial) + \mathbf{F}(\partial) \leq 3. \quad (\text{Vertex neutrosophic constraint})$$

(2) For every $\hat{\mathbf{g}} \in \mathbf{E}$,

$$0 \leq \top_{\mathbf{E}}(\hat{\mathbf{g}}) + \mathbf{l}_{\mathbf{E}}(\hat{\mathbf{g}}) + \mathbf{F}_{\mathbf{E}}(\hat{\mathbf{g}}) \leq 3. \quad (\text{Edge neutrosophic constraint})$$

(3) For every $\hat{\mathbf{g}} = \{\zeta, \partial\} \in \mathbf{E}$,

$$\top_{\mathbf{E}}(\hat{\mathbf{g}}) \leq \min\{\top(\zeta), \top(\partial)\},$$

$$\mathbf{l}_{\mathbf{E}}(\hat{\mathbf{g}}) \leq \max\{\mathbf{l}(\zeta), \mathbf{l}(\partial)\},$$

$$\mathbf{F}_{\mathbf{E}}(\hat{\mathbf{g}}) \geq \max\{\mathbf{F}(\zeta), \mathbf{F}(\partial)\}. \quad (\text{Membership consistency})$$

(4)

$$\xi(\partial, \hat{g}) = 0 \iff \partial \notin \hat{g}. \text{ (Incidence condition)}$$

(5) For every $\hat{g} = \{\zeta, \partial\} \in E$,

$$\xi(\zeta, \hat{g}), \xi(\partial, \hat{g}) \in \{-1, 1\}. \text{ (Local orientation)}$$

Example 3.1. Consider the crisp undirected simple graph $\mathcal{G}^* = (\mathcal{V}, E)$, where $V = \{\partial_1, \partial_2, \partial_3\}$ and $E = \{\{\partial_1, \partial_2\}, \{\partial_2, \partial_3\}, \{\partial_1, \partial_3\}\}$. We define the neutrosophic vertex functions by

$$\begin{aligned} \top(\partial_1) &= 0.8, \quad \text{I}(\partial_1) = 0.2, \quad \text{F}(\partial_1) = 0.1, \\ \top(\partial_2) &= 0.6, \quad \text{I}(\partial_2) = 0.3, \quad \text{F}(\partial_2) = 0.2, \\ \top(\partial_3) &= 0.7, \quad \text{I}(\partial_3) = 0.4, \quad \text{F}(\partial_3) = 0.2. \end{aligned}$$

Clearly,

$$0 \leq \top(\partial_i) + \text{I}(\partial_i) + \text{F}(\partial_i) \leq 3$$

for each $i = 1, 2, 3$, since

$$\top(\partial_1) + \text{I}(\partial_1) + \text{F}(\partial_1) = 1.1, \quad \top(\partial_2) + \text{I}(\partial_2) + \text{F}(\partial_2) = 1.1, \quad \text{and} \quad \top(\partial_3) + \text{I}(\partial_3) + \text{F}(\partial_3) = 1.3.$$

Now, we define the neutrosophic edge functions as follows:

$$\begin{aligned} \top_E(\{\partial_1, \partial_2\}) &= 0.5, \quad \text{I}_E(\{\partial_1, \partial_2\}) = 0.3, \quad \text{F}_E(\{\partial_1, \partial_2\}) = 0.2, \\ \top_E(\{\partial_2, \partial_3\}) &= 0.6, \quad \text{I}_E(\{\partial_2, \partial_3\}) = 0.4, \quad \text{F}_E(\{\partial_2, \partial_3\}) = 0.2, \\ \top_E(\{\partial_1, \partial_3\}) &= 0.7, \quad \text{I}_E(\{\partial_1, \partial_3\}) = 0.4, \quad \text{F}_E(\{\partial_1, \partial_3\}) = 0.2. \end{aligned}$$

Next, we now define the bidirection function

$$\xi : \mathcal{V} \times E \rightarrow \{-1, 0, 1\}$$

by

$$\begin{aligned} \xi(\partial_1, \{\partial_1, \partial_2\}) &= 1, \quad \xi(\partial_2, \{\partial_1, \partial_2\}) = -1, \\ \xi(\partial_2, \{\partial_2, \partial_3\}) &= 1, \quad \xi(\partial_3, \{\partial_2, \partial_3\}) = -1, \\ \xi(\partial_1, \{\partial_1, \partial_3\}) &= -1, \quad \xi(\partial_3, \{\partial_1, \partial_3\}) = 1. \end{aligned}$$

Also, $\xi(\partial, \hat{g}) = 0$ whenever $\partial \notin \hat{g}$.

Now, we verify the required conditions. For the edge $\{\partial_1, \partial_2\}$,

$$\begin{aligned} \top_E(\{\partial_1, \partial_2\}) &= 0.5 \leq \min\{0.8, 0.6\} = 0.6, \\ \text{I}_E(\{\partial_1, \partial_2\}) &= 0.3 \leq \max\{0.2, 0.3\} = 0.3, \end{aligned}$$

and

$$\text{F}_E(\{\partial_1, \partial_2\}) = 0.2 \geq \max\{0.1, 0.2\} = 0.2.$$

Similarly, for the edge $\{\partial_2, \partial_3\}$,

$$\begin{aligned} 0.6 &\leq \min\{0.6, 0.7\} = 0.6, \\ 0.4 &\leq \max\{0.3, 0.4\} = 0.4, \end{aligned}$$

and

$$0.2 \geq \max\{0.2, 0.2\} = 0.2.$$

For the edge $\{\partial_1, \partial_3\}$,

$$\begin{aligned} 0.7 &\leq \min\{0.8, 0.7\} = 0.7, \\ 0.4 &\leq \max\{0.2, 0.4\} = 0.4, \end{aligned}$$

and

$$0.2 \geq \max\{0.1, 0.2\} = 0.2.$$

Moreover,

$$0 \leq \top_E(\hat{g}) + \text{I}_E(\hat{g}) + \text{F}_E(\hat{g}) \leq 3$$

for every $\hat{g} \in E$.

The incidence condition holds because

$$\xi(\partial, \hat{g}) = 0 \iff \partial \notin \hat{g}.$$

Further, for every edge

$$\hat{g} = \{\zeta, \partial\} \in E,$$

we have

$$\xi(\zeta, \hat{g}), \xi(\partial, \hat{g}) \in \{-1, 1\},$$

which satisfies the local orientation condition. Hence, $\mathcal{B}_N = (\mathcal{T}, \mathcal{I}, \mathcal{F}, \mathcal{T}_E, \mathcal{I}_E, \mathcal{F}_E, \xi)$ is a neutrosophic bidirected graph.

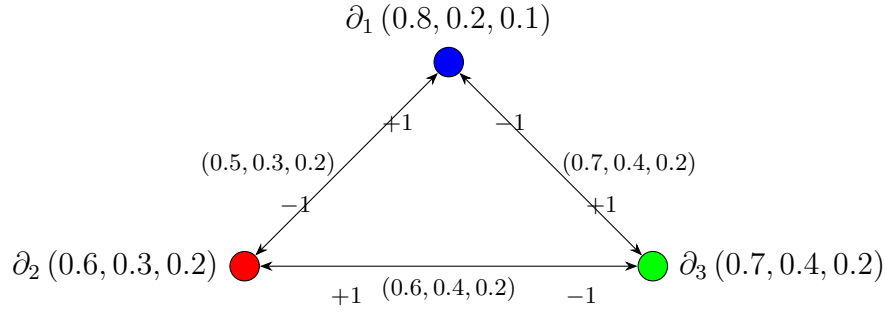


FIGURE 3. Neutrosophic bidirected graph corresponding to Example 3.1

Example 3.2. Consider the crisp undirected simple graph $\mathcal{G}^* = (\mathcal{V}, E)$, where $\mathcal{V} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ and

$$E = \{\{\alpha_1, \alpha_2\}, \{\alpha_2, \alpha_3\}, \{\alpha_3, \alpha_4\}, \{\alpha_1, \alpha_4\}\}.$$

Now, we define the neutrosophic vertex functions by

$$\mathcal{T}(\alpha_1) = 0.9, \quad \mathcal{I}(\alpha_1) = 0.1, \quad \mathcal{F}(\alpha_1) = 0.1,$$

$$\mathcal{T}(\alpha_2) = 0.7, \quad \mathcal{I}(\alpha_2) = 0.2, \quad \mathcal{F}(\alpha_2) = 0.2,$$

$$\mathcal{T}(\alpha_3) = 0.6, \quad \mathcal{I}(\alpha_3) = 0.3, \quad \mathcal{F}(\alpha_3) = 0.2,$$

$$\mathcal{T}(\alpha_4) = 0.8, \quad \mathcal{I}(\alpha_4) = 0.2, \quad \mathcal{F}(\alpha_4) = 0.1.$$

Next, define the neutrosophic edge functions by

$$\mathcal{T}_E(\{\alpha_1, \alpha_2\}) = 0.7, \quad \mathcal{I}_E(\{\alpha_1, \alpha_2\}) = 0.2, \quad \mathcal{F}_E(\{\alpha_1, \alpha_2\}) = 0.2,$$

$$\mathcal{T}_E(\{\alpha_2, \alpha_3\}) = 0.6, \quad \mathcal{I}_E(\{\alpha_2, \alpha_3\}) = 0.3, \quad \mathcal{F}_E(\{\alpha_2, \alpha_3\}) = 0.2,$$

$$\mathcal{T}_E(\{\alpha_3, \alpha_4\}) = 0.6, \quad \mathcal{I}_E(\{\alpha_3, \alpha_4\}) = 0.3, \quad \mathcal{F}_E(\{\alpha_3, \alpha_4\}) = 0.2,$$

$$\mathcal{T}_E(\{\alpha_1, \alpha_4\}) = 0.8, \quad \mathcal{I}_E(\{\alpha_1, \alpha_4\}) = 0.2, \quad \mathcal{F}_E(\{\alpha_1, \alpha_4\}) = 0.1.$$

Further, define the bidirection function

$$\xi : \mathcal{V} \times E \rightarrow \{-1, 0, 1\}$$

by

$$\xi(\alpha_1, \{\alpha_1, \alpha_2\}) = 1, \quad \xi(\alpha_2, \{\alpha_1, \alpha_2\}) = -1,$$

$$\xi(\alpha_2, \{\alpha_2, \alpha_3\}) = 1, \quad \xi(\alpha_3, \{\alpha_2, \alpha_3\}) = -1,$$

$$\xi(\alpha_3, \{\alpha_3, \alpha_4\}) = 1, \quad \xi(\alpha_4, \{\alpha_3, \alpha_4\}) = -1,$$

$$\xi(\alpha_1, \{\alpha_1, \alpha_4\}) = -1, \quad \xi(\alpha_4, \{\alpha_1, \alpha_4\}) = 1.$$

Hence, $\mathcal{B}_N = (\mathcal{T}, \mathcal{I}, \mathcal{F}, \mathcal{T}_E, \mathcal{I}_E, \mathcal{F}_E, \xi)$ is a neutrosophic bidirected graph.

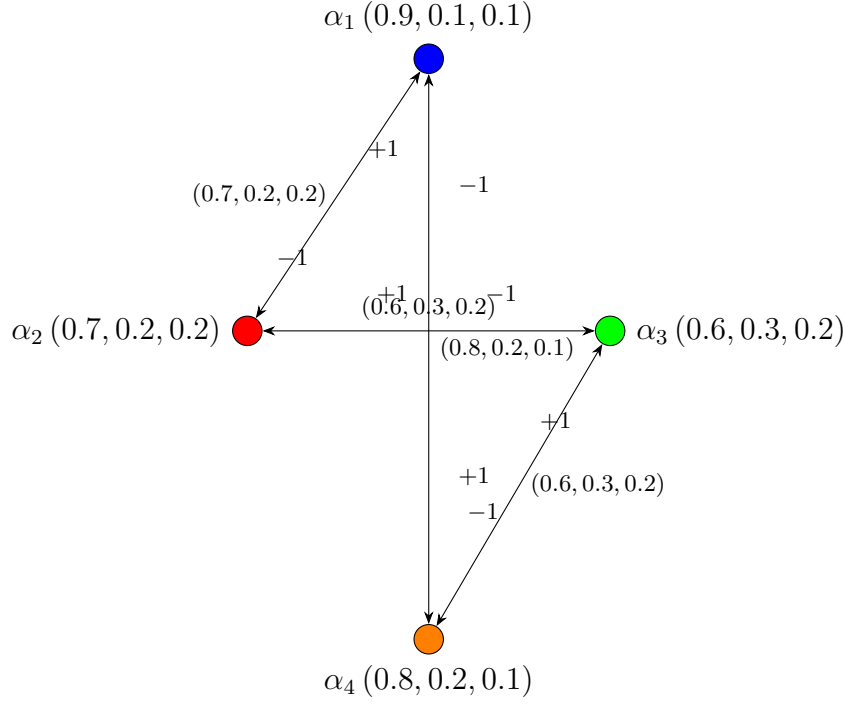


FIGURE 4. Neutrosophic bidirected graph corresponding to the above example 3.2

Theorem 3.1. Let $\mathcal{B}_N = (\top, \mathbf{l}, \mathbf{F}, \top_E, \mathbf{l}_E, \mathbf{F}_E, \xi)$ be a neutrosophic bidirected graph on $\mathcal{G}^* = (\mathcal{V}, \mathbf{E})$. Then:

- (1) If $\mathbf{l}(\partial) = 0$, $\mathbf{F}(\partial) = 0$ for all $\partial \in \mathcal{G}$, and $\mathbf{l}_E(\hat{\mathbf{g}}) = 0$, $\mathbf{F}_E(\hat{\mathbf{g}}) = 0$ for all $\hat{\mathbf{g}} \in \mathbf{E}$, then $\mathcal{B}_N$ reduces to a fuzzy bidirected graph.
- (2) If, in addition, $\top(\partial) = 1$ for all $\partial \in \mathcal{V}$, and $\top_E(\hat{\mathbf{g}}) = 1$ for all $\hat{\mathbf{g}} \in \mathbf{E}$, then $\mathcal{B}_N$ further reduces to a crisp bidirected graph.
- (3) If $\xi(\zeta, \hat{\mathbf{g}}) = -\xi(\partial, \hat{\mathbf{g}})$ for every edge $\hat{\mathbf{g}} = \{\zeta, \partial\} \in \mathbf{E}$. Then, $\mathcal{B}_N$ induces a neutrosophic directed graph on the digraph

$$\tilde{\mathbf{A}} = \left\{ (\iota_1, \iota_2) \mid \{\iota_1, \iota_2\} \in \mathbf{E}, \xi(\iota_1, \{\iota_1, \iota_2\}) = +1, \xi(\iota_2, \{\iota_1, \iota_2\}) = -1 \right\}.$$

Proof. (1) Setting $\mathbf{l} \equiv 0$, $\mathbf{F} \equiv 0$, and $\mathbf{l}_E \equiv 0$, $\mathbf{F}_E \equiv 0$ in $\mathcal{B}_N$ leaves exactly the data $(\top, \top_E, \xi)$, satisfying the axioms of a fuzzy bidirected graph.

(2) Further imposing $\top \equiv 1$, $\top_E \equiv 1$ makes all membership consistency conditions trivial, recovering the usual crisp bidirected graph $(\mathcal{G}^*, \xi)$.

(3) Assume $\xi(\zeta, \hat{\mathbf{g}}) = -\xi(\partial, \hat{\mathbf{g}})$ for every edge $\hat{\mathbf{g}} = \{\zeta, \partial\} \in \mathbf{E}$.

we define

$$\tilde{\mathbf{A}} = \left\{ (\iota_1, \iota_2) \in \mathcal{V} \times \mathcal{V} : \{\iota_1, \iota_2\} \in \mathbf{E}, \xi(\iota_1, \{\iota_1, \iota_2\}) = +1, \xi(\iota_2, \{\iota_1, \iota_2\}) = -1 \right\}.$$

Then, $\vec{\mathcal{G}} = (\mathcal{V}, \tilde{\mathbf{A}})$ is a crisp directed graph. Define

$$\top_{\tilde{\mathbf{A}}}(\iota_1, \iota_2) = \top_E(\{\iota_1, \iota_2\}),$$

$$\mathbf{l}_{\tilde{\mathbf{A}}}(\iota_1, \iota_2) = \mathbf{l}_E(\{\iota_1, \iota_2\}),$$

and

$$\mathbf{F}_{\tilde{\mathbf{A}}}(\iota_1, \iota_2) = \mathbf{F}_E(\{\iota_1, \iota_2\}),$$

and retain the vertex functions $\top$, $\mathbf{l}$, $\mathbf{F}$. Since

$$\top_E(\hat{\mathbf{g}}) \leq \min\{\top(\zeta), \top(\partial)\},$$

$$\mathbf{l}_E(\hat{\mathbf{g}}) \leq \max\{\mathbf{l}(\zeta), \mathbf{l}(\partial)\},$$

and

$$F_E(\hat{g}) \geq \max\{F(\zeta), F(\partial)\}$$

hold in $\mathcal{B}_N$, it follows that

$$\begin{aligned} \top_{\tilde{A}}(\iota_1, \iota_2) &\leq \min\{\top(\iota_1), \top(\iota_2)\}, \\ I_{\tilde{A}}(\iota_1, \iota_2) &\leq \max\{I(\iota_1), I(\iota_2)\}, \end{aligned}$$

and

$$F_{\tilde{A}}(\iota_1, \iota_2) \geq \max\{F(\iota_1), F(\iota_2)\}.$$

Moreover,

$$0 \leq \top_{\tilde{A}}(\iota_1, \iota_2) + I_{\tilde{A}}(\iota_1, \iota_2) + F_{\tilde{A}}(\iota_1, \iota_2) \leq 3.$$

Hence,

$$(\top, I, F, \top_{\tilde{A}}, I_{\tilde{A}}, F_{\tilde{A}})$$

satisfies the axioms of a neutrosophic directed graph on $\vec{\mathcal{G}}$. $\square$

Theorem 3.2. Let $\mathcal{B}_{N_1} = (\top_1, I_1, F_1, \top_{E_1}, I_{E_1}, F_{E_1}, \xi)$ and $\mathcal{B}_{N_2} = (\top_2, I_2, F_2, \top_{E_2}, I_{E_2}, F_{E_2}, \xi)$ be two neutrosophic bidirected graphs on the same crisp graph $\mathcal{G}^* = (\mathcal{V}, E)$. Define $\top = \top_1 \cap \top_2$, $I = I_1 \cup I_2$, and $F = F_1 \cup F_2$, and similarly for edge functions. Then, $(\top, I, F, \top_E, I_E, F_E, \xi)$ is also a neutrosophic bidirected graph on $\mathcal{G}^*$.

Proof. We verify each axiom.

1. Vertex neutrosophic constraint. Fix $\partial \in \mathcal{V}$. Since

$$0 \leq \top_i(\partial) + I_i(\partial) + F_i(\partial) \leq 3 \quad \text{for } i = 1, 2,$$

we obtain

$$\top(\partial) + I(\partial) + F(\partial) = \min\{\top_1(\partial), \top_2(\partial)\} + \max\{I_1(\partial), I_2(\partial)\} + \max\{F_1(\partial), F_2(\partial)\}.$$

Clearly,

$$0 \leq \top(\partial) + I(\partial) + F(\partial) \leq 3.$$

2. Edge neutrosophic constraint. Similarly, for every $\hat{g} \in E$,

$$0 \leq \top_E(\hat{g}) + I_E(\hat{g}) + F_E(\hat{g}) \leq 3.$$

3. Membership consistency. For any edge

$$\hat{g} = \{\zeta, \partial\} \in E,$$

we have

$$\top_E(\hat{g}) = \min\{\top_{E_1}(\hat{g}), \top_{E_2}(\hat{g})\}.$$

Since

$$\top_{E_i}(\hat{g}) \leq \min\{\top_i(\zeta), \top_i(\partial)\}$$

for $i = 1, 2$, it follows that

$$\top_E(\hat{g}) \leq \min\{\top(\zeta), \top(\partial)\}.$$

Also,

$$I_E(\hat{g}) = \max\{I_{E_1}(\hat{g}), I_{E_2}(\hat{g})\}$$

and

$$I_{E_i}(\hat{g}) \leq \max\{I_i(\zeta), I_i(\partial)\}.$$

Thus,

$$I_E(\hat{g}) \leq \max\{I(\zeta), I(\partial)\}.$$

Likewise,

$$F_E(\hat{g}) = \max\{F_{E_1}(\hat{g}), F_{E_2}(\hat{g})\}$$

and

$$F_{E_i}(\hat{g}) \geq \max\{F_i(\zeta), F_i(\partial)\}.$$

Hence,

$$F_E(\hat{g}) \geq \max\{F(\zeta), F(\partial)\}.$$

4. Incidence condition. Since the bidirection function ξ is unchanged,

$$\xi(\partial, \hat{g}) = 0 \iff \partial \notin \hat{g}$$

still holds.

5. Local orientation. Similarly,

$$\xi(\zeta, \hat{g}), \xi(\partial, \hat{g}) \in \{-1, 1\}$$

for every edge

$$\hat{g} = \{\zeta, \partial\} \in E.$$

Therefore,

$$(\top, I, F, \top_E, I_E, F_E, \xi)$$

satisfies all axioms of a neutrosophic bidirected graph on $\mathcal{G}^*$. $\square$

Theorem 3.3 (Union of Neutrosophic Bidirected Graphs). *Let $\mathcal{B}_{N_1} = (\top_1, I_1, F_1, \top_{E_1}, I_{E_1}, F_{E_1}, \xi)$ and $\mathcal{B}_{N_2} = (\top_2, I_2, F_2, \top_{E_2}, I_{E_2}, F_{E_2}, \xi)$ be two neutrosophic bidirected graphs on the same underlying crisp graph $\mathcal{G}^* = (\mathcal{V}, E)$, sharing the same bidirection function ξ .*

Define

$$\top(\partial) = \max\{\top_1(\partial), \top_2(\partial)\},$$

$$I(\partial) = \min\{I_1(\partial), I_2(\partial)\},$$

$$F(\partial) = \min\{F_1(\partial), F_2(\partial)\},$$

and

$$\top_E(\hat{g}) = \max\{\top_{E_1}(\hat{g}), \top_{E_2}(\hat{g})\},$$

$$I_E(\hat{g}) = \min\{I_{E_1}(\hat{g}), I_{E_2}(\hat{g})\},$$

$$F_E(\hat{g}) = \min\{F_{E_1}(\hat{g}), F_{E_2}(\hat{g})\}.$$

Then, $(\top, I, F, \top_E, I_E, F_E, \xi)$ is again a neutrosophic bidirected graph on $\mathcal{G}^*$.

Proof. We verify the defining axioms.

1. Vertex neutrosophic constraint. For each $\partial \in \mathcal{V}$,

$$\top(\partial) + I(\partial) + F(\partial) = \max\{\top_1(\partial), \top_2(\partial)\} + \min\{I_1(\partial), I_2(\partial)\} + \min\{F_1(\partial), F_2(\partial)\}.$$

Since

$$0 \leq \top_i(\partial) + I_i(\partial) + F_i(\partial) \leq 3$$

for $i = 1, 2$, it follows immediately that

$$0 \leq \top(\partial) + I(\partial) + F(\partial) \leq 3.$$

2. Edge neutrosophic constraint. Similarly, for every $\hat{g} \in E$,

$$0 \leq \top_E(\hat{g}) + I_E(\hat{g}) + F_E(\hat{g}) \leq 3.$$

3. Membership consistency. Let

$$\hat{g} = \{\zeta, \partial\} \in E.$$

Since

$$\top_{E_i}(\hat{g}) \leq \min\{\top_i(\zeta), \top_i(\partial)\}$$

for $i = 1, 2$, we obtain

$$\top_E(\hat{g}) = \max\{\top_{E_1}(\hat{g}), \top_{E_2}(\hat{g})\} \leq \min\{\top(\zeta), \top(\partial)\}.$$

Also,

$$I_{E_i}(\hat{g}) \leq \max\{I_i(\zeta), I_i(\partial)\}.$$

Hence,

$$I_E(\hat{g}) = \min\{I_{E_1}(\hat{g}), I_{E_2}(\hat{g})\} \leq \max\{I(\zeta), I(\partial)\}.$$

Likewise,

$$F_{E_i}(\hat{g}) \geq \max\{F_i(\zeta), F_i(\partial)\},$$

therefore

$$F_E(\hat{g}) = \min\{F_{E_1}(\hat{g}), F_{E_2}(\hat{g})\} \geq \max\{F(\zeta), F(\partial)\}.$$

4. Incidence condition. Since the bidirection function ξ remains unchanged,

$$\xi(\partial, \hat{g}) = 0 \iff \partial \notin \hat{g}.$$

5. Local orientation. For every edge

$$\hat{g} = \{\zeta, \partial\} \in E,$$

we still have

$$\xi(\zeta, \hat{g}), \xi(\partial, \hat{g}) \in \{-1, 1\}.$$

Hence,

$$(\top, I, F, \top_E, I_E, F_E, \xi)$$

satisfies all axioms of a neutrosophic bidirected graph on $\mathcal{G}^*$. $\square$

Theorem 3.4. Let $\mathcal{B}_N = (\top, I, F, \top_E, I_E, F_E, \xi)$ be a neutrosophic bidirected graph on $\mathcal{G}^* = (\mathcal{V}, E)$. Define the complement $\mathcal{B}_N^c = (\top^c, I^c, F^c, \top_E^c, I_E^c, F_E^c, \xi)$, where for every $\partial \in \mathcal{V}$ and $\hat{g} \in E$, $\top^c(\partial) = F(\partial)$, $I^c(\partial) = I(\partial)$, $F^c(\partial) = \top(\partial)$, and $\top_E^c(\hat{g}) = F_E(\hat{g})$, $I_E^c(\hat{g}) = I_E(\hat{g})$, $F_E^c(\hat{g}) = \top_E(\hat{g})$. Then, $\mathcal{B}_N^c$ is also a neutrosophic bidirected graph.

Proof. For every $\partial \in \mathcal{V}$,

$$0 \leq \top(\partial) + I(\partial) + F(\partial) \leq 3.$$

Hence,

$$0 \leq \top^c(\partial) + I^c(\partial) + F^c(\partial) \leq 3.$$

Similarly,

$$0 \leq \top_E^c(\hat{g}) + I_E^c(\hat{g}) + F_E^c(\hat{g}) \leq 3.$$

Now,

$$\top_E^c(\hat{g}) = F_E(\hat{g}) \leq \min\{F(\zeta), F(\partial)\} = \min\{\top^c(\zeta), \top^c(\partial)\}.$$

Also,

$$I_E^c(\hat{g}) = I_E(\hat{g}) \leq \max\{I(\zeta), I(\partial)\} = \max\{I^c(\zeta), I^c(\partial)\}.$$

Likewise,

$$F_E^c(\hat{g}) = \top_E(\hat{g}) \geq \max\{\top(\zeta), \top(\partial)\} = \max\{F^c(\zeta), F^c(\partial)\}.$$

The incidence and orientation conditions remain unchanged since ξ is preserved. Therefore $\mathcal{B}_N^c$ is a neutrosophic bidirected graph. $\square$

Theorem 3.5. Let $\mathcal{B}_N = (\top, I, F, \top_E, I_E, F_E, \xi)$ be a neutrosophic bidirected graph. If for every edge $\hat{g} = \{\zeta, \partial\} \in E$, the equalities

$$\top_E(\hat{g}) = \min\{\top(\zeta), \top(\partial)\}, I_E(\hat{g}) = \max\{I(\zeta), I(\partial)\}, \text{ and } F_E(\hat{g}) = \max\{F(\zeta), F(\partial)\}$$

hold. Then, $\mathcal{B}_N$ is called a strong neutrosophic bidirected graph.

Proof. The result follows directly from the defining equalities, which strengthen the membership consistency conditions of a neutrosophic bidirected graph. $\square$

Theorem 3.6. Let $\mathcal{B}_{N_1} = (\top_1, I_1, F_1, \top_{E_1}, I_{E_1}, F_{E_1}, \xi_1)$ and $\mathcal{B}_{N_2} = (\top_2, I_2, F_2, \top_{E_2}, I_{E_2}, F_{E_2}, \xi_2)$ be neutrosophic bidirected graphs. Suppose $\phi : \partial_1 \rightarrow \partial_2$ is a graph homomorphism satisfying $\top_1(\partial) \leq \top_2(\phi(\partial))$, $I_1(\partial) \geq I_2(\phi(\partial))$, and $F_1(\partial) \geq F_2(\phi(\partial))$ for all $\partial \in \partial_1$. Then, ϕ preserves the neutrosophic bidirected structure.

Proof. Let $\hat{g} = \{\zeta, \partial\} \in E_1$. Since $\phi : \partial_1 \rightarrow \partial_2$ is a graph homomorphism, we have

$$\phi(\hat{g}) = \{\phi(\zeta), \phi(\partial)\} \in E_2.$$

Now, because $\mathcal{B}_{N_1}$ is a neutrosophic bidirected graph, the edge membership functions satisfy

$$\begin{aligned} \top_{E_1}(\hat{g}) &\leq \min\{\top_1(\zeta), \top_1(\partial)\}, \\ \mathbf{l}_{E_1}(\hat{g}) &\leq \max\{\mathbf{l}_1(\zeta), \mathbf{l}_1(\partial)\}, \end{aligned}$$

and

$$\mathbf{F}_{E_1}(\hat{g}) \geq \max\{\mathbf{F}_1(\zeta), \mathbf{F}_1(\partial)\}.$$

Using the assumptions

$$\begin{aligned} \top_1(\iota_1) &\leq \top_2(\phi(\iota_1)), \\ \mathbf{l}_1(\iota_1) &\geq \mathbf{l}_2(\phi(\iota_1)), \end{aligned}$$

and

$$\mathbf{F}_1(\iota_1) \geq \mathbf{F}_2(\phi(\iota_1))$$

for every $\iota_1 \in \partial_1$, we obtain

$$\min\{\top_1(\zeta), \top_1(\partial)\} \leq \min\{\top_2(\phi(\zeta)), \top_2(\phi(\partial))\}.$$

Hence,

$$\top_{E_1}(\hat{g}) \leq \min\{\top_2(\phi(\zeta)), \top_2(\phi(\partial))\}.$$

Similarly,

$$\max\{\mathbf{l}_1(\zeta), \mathbf{l}_1(\partial)\} \geq \max\{\mathbf{l}_2(\phi(\zeta)), \mathbf{l}_2(\phi(\partial))\}.$$

Therefore,

$$\mathbf{l}_{E_1}(\hat{g}) \leq \max\{\mathbf{l}_2(\phi(\zeta)), \mathbf{l}_2(\phi(\partial))\}.$$

Also,

$$\max\{\mathbf{F}_1(\zeta), \mathbf{F}_1(\partial)\} \geq \max\{\mathbf{F}_2(\phi(\zeta)), \mathbf{F}_2(\phi(\partial))\},$$

which implies

$$\mathbf{F}_{E_1}(\hat{g}) \geq \max\{\mathbf{F}_2(\phi(\zeta)), \mathbf{F}_2(\phi(\partial))\}.$$

Moreover, since $\mathcal{B}_{N_2}$ is a neutrosophic bidirected graph,

$$0 \leq \top_2(\phi(\partial)) + \mathbf{l}_2(\phi(\partial)) + \mathbf{F}_2(\phi(\partial)) \leq 3$$

for all $\partial \in \partial_1$, and

$$0 \leq \top_{E_2}(\phi(\hat{g})) + \mathbf{l}_{E_2}(\phi(\hat{g})) + \mathbf{F}_{E_2}(\phi(\hat{g})) \leq 3.$$

The bidirection and incidence conditions are preserved under ϕ , since adjacency and edge incidences are preserved by graph homomorphisms.

Therefore, ϕ preserves the neutrosophic bidirected structure from $\mathcal{B}_{N_1}$ to $\mathcal{B}_{N_2}$. $\square$

Theorem 3.7. Let $\mathcal{B}_{N_1} = (\top_1, \mathbf{l}_1, \mathbf{F}_1, \top_{E_1}, \mathbf{l}_{E_1}, \mathbf{F}_{E_1}, \xi_1)$ and $\mathcal{B}_{N_2} = (\top_2, \mathbf{l}_2, \mathbf{F}_2, \top_{E_2}, \mathbf{l}_{E_2}, \mathbf{F}_{E_2}, \xi_2)$ be two neutrosophic bidirected graphs. If $\phi : \partial_1 \rightarrow \partial_2$ preserves all vertex and edge neutrosophic values, then $\mathcal{B}_{N_1} \cong \mathcal{B}_{N_2}$.

Proof. Since ϕ is a graph isomorphism, it is bijective and preserves adjacency as well as incidences between vertices and edges.

Let $\hat{g} = \{\zeta, \partial\} \in E_1$. Then,

$$\phi(\hat{g}) = \{\phi(\zeta), \phi(\partial)\} \in E_2.$$

Using the assumptions,

$$\top_{E_1}(\hat{g}) = \top_{E_2}(\phi(\hat{g})),$$

and

$$\top_1(\zeta) = \top_2(\phi(\zeta)), \quad \top_1(\partial) = \top_2(\phi(\partial)).$$

Since $\mathcal{B}_{N_1}$ satisfies

$$\top_{E_1}(\hat{g}) \leq \min\{\top_1(\zeta), \top_1(\partial)\},$$

we obtain

$$\top_{E_2}(\phi(\hat{g})) \leq \min\{\top_2(\phi(\zeta)), \top_2(\phi(\partial))\}.$$

Similarly,

$$I_{E_2}(\phi(\hat{g})) \leq \max\{I_2(\phi(\zeta)), I_2(\phi(\partial))\},$$

and

$$F_{E_2}(\phi(\hat{g})) \geq \max\{F_2(\phi(\zeta)), F_2(\phi(\partial))\}.$$

Thus, all neutrosophic consistency conditions are preserved under ϕ .

Moreover, because ϕ preserves incidences and orientations,

$$\xi_1(\partial, \hat{g}) = \xi_2(\phi(\partial), \phi(\hat{g})).$$

Therefore, $\mathcal{B}_{N_1} \cong \mathcal{B}_{N_2}$, and hence, the two neutrosophic bidirected graphs are isomorphic. $\square$

Theorem 3.8. *Let $\mathcal{B}_N = (\top, I, F, \top_E, I_E, F_E, \xi)$ be a neutrosophic bidirected graph. If $V' \subseteq V$ and $E' = \{\{\zeta, \partial\} \in E : \zeta, \partial \in V'\} \subseteq E$. Then, the restriction $\mathcal{B}'_N = (\top', I', F', \top'_E, I'_E, F'_E, \xi')$ forms a neutrosophic bidirected subgraph of $\mathcal{B}_N$.*

Proof. Since $V' \subseteq V$ and $E' \subseteq E$, all vertex and edge neutrosophic values are inherited directly from $\mathcal{B}_N$. For every vertex $\partial \in V'$, because $\mathcal{B}_N$ is a neutrosophic bidirected graph,

$$0 \leq \top(\partial) + I(\partial) + F(\partial) \leq 3.$$

Hence,

$$0 \leq \top'(\partial) + I'(\partial) + F'(\partial) \leq 3.$$

Similarly, for every edge $\hat{g} \in E'$, we have

$$0 \leq \top_E(\hat{g}) + I_E(\hat{g}) + F_E(\hat{g}) \leq 3,$$

which implies

$$0 \leq \top'_E(\hat{g}) + I'_E(\hat{g}) + F'_E(\hat{g}) \leq 3.$$

Now, let

$$\hat{g} = \{\zeta, \partial\} \in E'.$$

Since $\hat{g} \in E$, the membership consistency conditions in $\mathcal{B}_N$ yield

$$\top_E(\hat{g}) \leq \min\{\top(\zeta), \top(\partial)\},$$

$$I_E(\hat{g}) \leq \max\{I(\zeta), I(\partial)\},$$

and

$$F_E(\hat{g}) \geq \max\{F(\zeta), F(\partial)\}.$$

Using the definitions of the restricted functions,

$$\top'_E(\hat{g}) \leq \min\{\top'(\zeta), \top'(\partial)\},$$

$$I'_E(\hat{g}) \leq \max\{I'(\zeta), I'(\partial)\},$$

and

$$F'_E(\hat{g}) \geq \max\{F'(\zeta), F'(\partial)\}.$$

Furthermore, since ξ' is the restriction of ξ ,

$$\xi'(\partial, \hat{g}) = 0 \iff \partial \notin \hat{g},$$

and for every

$$\hat{g} = \{\zeta, \partial\} \in E',$$

$$\xi'(\zeta, \hat{g}), \xi'(\partial, \hat{g}) \in \{-1, 1\}.$$

Therefore, $\mathcal{B}'_N$ satisfies all axioms of a neutrosophic bidirected graph and hence forms a neutrosophic bidirected subgraph of $\mathcal{B}_N$. $\square$

3.1. Structural Properties of Strong Neutrosophic Bidirected Graphs.

Theorem 3.9. *Let $\mathcal{B}_N = (\mathbb{T}, \mathbb{I}, \mathbb{F}, \mathbb{T}_E, \mathbb{I}_E, \mathbb{F}_E, \xi)$ be a strong neutrosophic bidirected graph. If $\mathcal{B}'_N$ is an induced neutrosophic bidirected subgraph of $\mathcal{B}_N$, then $\mathcal{B}'_N$ is also a strong neutrosophic bidirected graph.*

Proof. Let

$$V' \subseteq V$$

and let

$$E' = \{\{\zeta, \partial\} \in E : \zeta, \partial \in V'\}$$

be the edge set of the induced subgraph. Since $\mathcal{B}'_N$ is induced, all vertex and edge neutrosophic values are inherited from $\mathcal{B}_N$. Take any edge

$$\hat{g} = \{\zeta, \partial\} \in E'.$$

Since $\mathcal{B}_N$ is strong,

$$\mathbb{T}_E(\hat{g}) = \min\{\mathbb{T}(\zeta), \mathbb{T}(\partial)\}, \mathbb{I}_E(\hat{g}) = \max\{\mathbb{I}(\zeta), \mathbb{I}(\partial)\}, \text{ and } \mathbb{F}_E(\hat{g}) = \max\{\mathbb{F}(\zeta), \mathbb{F}(\partial)\}.$$

Because the induced subgraph preserves these values,

$$\mathbb{T}'_E(\hat{g}) = \min\{\mathbb{T}'(\zeta), \mathbb{T}'(\partial)\}, \mathbb{I}'_E(\hat{g}) = \max\{\mathbb{I}'(\zeta), \mathbb{I}'(\partial)\}, \text{ and } \mathbb{F}'_E(\hat{g}) = \max\{\mathbb{F}'(\zeta), \mathbb{F}'(\partial)\}.$$

Hence, every edge of $\mathcal{B}'_N$ satisfies the defining equalities of a strong neutrosophic bidirected graph. Therefore, $\mathcal{B}'_N$ is strong. $\square$

Theorem 3.10. *Let $\mathcal{B}_{N_1} \cong \mathcal{B}_{N_2}$. Then, $\mathcal{B}_{N_1}$ is strong if and only if $\mathcal{B}_{N_2}$ is strong.*

Proof. Let $\phi : V_1 \rightarrow V_2$ be an isomorphism between $\mathcal{B}_{N_1}$ and $\mathcal{B}_{N_2}$.

Assume that $\mathcal{B}_{N_1}$ is strong.

Take any edge

$$\{\phi(\zeta), \phi(\partial)\} \in E_2.$$

Since ϕ is an isomorphism,

$$\{\zeta, \partial\} \in E_1$$

and all neutrosophic values are preserved. Thus,

$$\mathbb{T}_2(\phi(\zeta)) = \mathbb{T}_1(\zeta), \quad \mathbb{T}_2(\phi(\partial)) = \mathbb{T}_1(\partial),$$

and similarly for the indeterminacy and falsity memberships. Since $\mathcal{B}_{N_1}$ is strong,

$$\mathbb{T}_{E_1}(\{\zeta, \partial\}) = \min\{\mathbb{T}_1(\zeta), \mathbb{T}_1(\partial)\}.$$

Applying ϕ ,

$$\mathbb{T}_{E_2}(\{\phi(\zeta), \phi(\partial)\}) = \min\{\mathbb{T}_2(\phi(\zeta)), \mathbb{T}_2(\phi(\partial))\}.$$

Similarly,

$$\mathbb{I}_{E_2} = \max\{\mathbb{I}_2(\phi(\zeta)), \mathbb{I}_2(\phi(\partial))\},$$

and

$$\mathbb{F}_{E_2} = \max\{\mathbb{F}_2(\phi(\zeta)), \mathbb{F}_2(\phi(\partial))\}.$$

Hence, $\mathcal{B}_{N_2}$ is strong.

The converse follows by applying the same argument to ϕ^{-1} . Therefore, strongness is preserved under isomorphism. $\square$

Theorem 3.11. *Let*

$$\mathcal{B}_N = (\mathbb{T}, \mathbb{I}, \mathbb{F}, \mathbb{T}_E, \mathbb{I}_E, \mathbb{F}_E, \xi)$$

be a neutrosophic bidirected graph. Define

$$\widetilde{\mathbb{T}}_E(\{\zeta, \partial\}) = \min\{\mathbb{T}(\zeta), \mathbb{T}(\partial)\},$$

$$\widetilde{\mathbb{I}}_E(\{\zeta, \partial\}) = \max\{\mathbb{I}(\zeta), \mathbb{I}(\partial)\},$$

and

$$\tilde{F}_E(\{\zeta, \partial\}) = \max\{F(\zeta), F(\partial)\}.$$

Then

$$\tilde{\mathcal{B}}_N = (\top, \text{I}, F, \tilde{\top}_E, \tilde{\text{I}}_E, \tilde{F}_E, \xi)$$

is a strong neutrosophic bidirected graph.

Proof. For every edge

$$\hat{g} = \{\zeta, \partial\} \in E,$$

the definitions immediately give

$$\tilde{\top}_E(\hat{g}) = \min\{\top(\zeta), \top(\partial)\},$$

$$\tilde{\text{I}}_E(\hat{g}) = \max\{\text{I}(\zeta), \text{I}(\partial)\},$$

and

$$\tilde{F}_E(\hat{g}) = \max\{F(\zeta), F(\partial)\}.$$

Hence, the membership consistency conditions hold with equality. Moreover,

$$0 \leq \tilde{\top}_E, \tilde{\text{I}}_E, \tilde{F}_E \leq 1,$$

because minima and maxima of numbers in $[0, 1]$ remain in $[0, 1]$.

The bidirection function ξ is unchanged, so the incidence and local orientation conditions remain valid. Therefore

$$\tilde{\mathcal{B}}_N$$

satisfies all requirements of a neutrosophic bidirected graph and, by construction, satisfies the defining equalities of a strong neutrosophic bidirected graph.

Hence,

$$\tilde{\mathcal{B}}_N$$

is strong. □

Example 3.3. Consider the neutrosophic bidirected graph

$$\mathcal{B}_N = (\top, \text{I}, F, \top_E, \text{I}_E, F_E, \xi)$$

on the crisp graph

$$G^* = (V, E), \quad V = \{v_1, v_2\}, \quad E = \{\hat{g}\},$$

where

$$\hat{g} = \{v_1, v_2\}.$$

Let

$$(\top(v_1), \text{I}(v_1), F(v_1)) = (0.9, 0.1, 0.2),$$

$$(\top(v_2), \text{I}(v_2), F(v_2)) = (0.7, 0.3, 0.4).$$

Assume that

$$(\top_E(\hat{g}), \text{I}_E(\hat{g}), F_E(\hat{g})) = (0.5, 0.2, 0.4).$$

Since

$$\min\{0.9, 0.7\} = 0.7 \neq 0.5,$$

the graph is not strong.

Now define

$$\tilde{\top}_E(\hat{g}) = \min\{0.9, 0.7\} = 0.7,$$

$$\tilde{\text{I}}_E(\hat{g}) = \max\{0.1, 0.3\} = 0.3,$$

and

$$\tilde{F}_E(\hat{g}) = \max\{0.2, 0.4\} = 0.4.$$

Hence

$$(\tilde{\top}_E, \tilde{\text{I}}_E, \tilde{F}_E) = (0.7, 0.3, 0.4).$$

Therefore

$$\tilde{\mathcal{B}}_N = (\top, \text{I}, \text{F}, \tilde{\top}_E, \tilde{\text{I}}_E, \tilde{\text{F}}_E, \xi)$$

satisfies

$$\begin{aligned}\tilde{\top}_E(\hat{g}) &= \min\{\top(v_1), \top(v_2)\}, \\ \tilde{\text{I}}_E(\hat{g}) &= \max\{\text{I}(v_1), \text{I}(v_2)\},\end{aligned}$$

and

$$\tilde{\text{F}}_E(\hat{g}) = \max\{\text{F}(v_1), \text{F}(v_2)\}.$$

Thus

$$\tilde{\mathcal{B}}_N$$

is a strong neutrosophic bidirected graph.

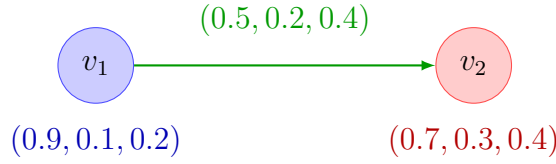


FIGURE 5. The original neutrosophic bidirected graph $\mathcal{B}_N$, which is not strong.

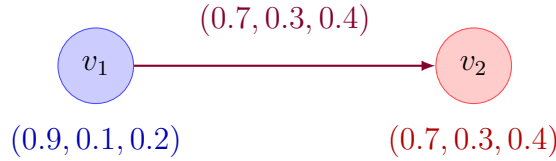


FIGURE 6. The canonical strong extension $\tilde{\mathcal{B}}_N$.

Remark 3.1. Observe that the vertex neutrosophic values and the bidirection function remain unchanged. Only the edge neutrosophic values are replaced by their extremal admissible values, thereby converting $\mathcal{B}_N$ into a strong neutrosophic bidirected graph.

Theorem 3.12. Let $\mathcal{B}_N = (\top, \text{I}, \text{F}, \top_E, \text{I}_E, \text{F}_E, \xi)$ be a neutrosophic bidirected graph.

Define $\xi^-(\partial, \hat{g}) = -\xi(\partial, \hat{g})$ for every $\partial \in V$, $\hat{g} \in E$. Then, $\mathcal{B}_N^- = (\top, \text{I}, \text{F}, \top_E, \text{I}_E, \text{F}_E, \xi^-)$ is also a neutrosophic bidirected graph.

Proof. Observe that the orientation reversal changes only the bidirection function ξ . The vertex memberships

$$\top, \quad \text{I}, \quad \text{F},$$

and the edge memberships

$$\top_E, \quad \text{I}_E, \quad \text{F}_E$$

remain unchanged.

Therefore, the vertex neutrosophic constraint, the edge neutrosophic constraint, and the membership consistency conditions continue to hold for $\mathcal{B}_N^-$.

It remains to verify the incidence and local orientation conditions.

Let $\partial \in V$ and $\hat{g} \in E$. Then

$$\xi^-(\partial, \hat{g}) = 0 \iff -\xi(\partial, \hat{g}) = 0 \iff \xi(\partial, \hat{g}) = 0.$$

Since $\mathcal{B}_N$ is a neutrosophic bidirected graph,

$$\xi(\partial, \hat{g}) = 0 \iff \partial \notin \hat{g}.$$

Hence

$$\xi^-(\partial, \hat{\mathbf{g}}) = 0 \iff \partial \notin \hat{\mathbf{g}},$$

so the incidence condition is preserved.

Now let

$$\hat{\mathbf{g}} = \{\zeta, \partial\} \in E.$$

Because $\mathcal{B}_N$ satisfies the local orientation condition,

$$\xi(\zeta, \hat{\mathbf{g}}), \xi(\partial, \hat{\mathbf{g}}) \in \{-1, 1\}.$$

Multiplying by -1 yields

$$\xi^-(\zeta, \hat{\mathbf{g}}), \xi^-(\partial, \hat{\mathbf{g}}) \in \{-1, 1\}.$$

Thus the local orientation condition also remains valid.

Consequently, all defining axioms of a neutrosophic bidirected graph are satisfied by

$$\mathcal{B}_N^-.$$

Therefore,

$$\mathcal{B}_N^-$$

is a neutrosophic bidirected graph. □

Example 3.4. Consider the neutrosophic bidirected graph

$$\mathcal{B}_N = (\top, \mathbf{l}, \mathbf{F}, \top_E, \mathbf{l}_E, \mathbf{F}_E, \xi)$$

on the crisp graph

$$G^* = (V, E), \quad V = \{v_1, v_2\}, \quad E = \{\hat{\mathbf{g}}\},$$

where

$$\hat{\mathbf{g}} = \{v_1, v_2\}.$$

Assign the vertex neutrosophic values

$$(\top(v_1), \mathbf{l}(v_1), \mathbf{F}(v_1)) = (0.8, 0.2, 0.1),$$

$$(\top(v_2), \mathbf{l}(v_2), \mathbf{F}(v_2)) = (0.6, 0.3, 0.2).$$

For the edge $\hat{\mathbf{g}}$, let

$$(\top_E(\hat{\mathbf{g}}), \mathbf{l}_E(\hat{\mathbf{g}}), \mathbf{F}_E(\hat{\mathbf{g}})) = (0.6, 0.3, 0.2).$$

Define the bidirection function by

$$\xi(v_1, \hat{\mathbf{g}}) = 1, \quad \xi(v_2, \hat{\mathbf{g}}) = -1.$$

Hence the edge is oriented from v_1 toward v_2 .

Now define

$$\xi^-(v_i, \hat{\mathbf{g}}) = -\xi(v_i, \hat{\mathbf{g}}), \quad i = 1, 2.$$

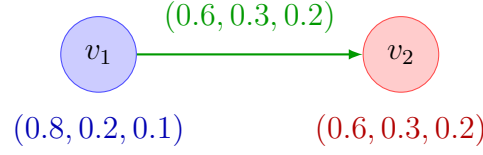
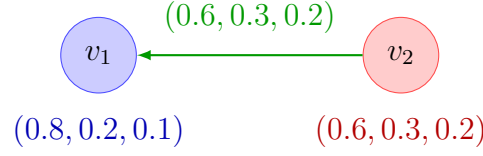
Then

$$\xi^-(v_1, \hat{\mathbf{g}}) = -1, \quad \xi^-(v_2, \hat{\mathbf{g}}) = 1.$$

Thus the orientation of the edge is reversed while all neutrosophic vertex and edge values remain unchanged. Therefore

$$\mathcal{B}_N^- = (\top, \mathbf{l}, \mathbf{F}, \top_E, \mathbf{l}_E, \mathbf{F}_E, \xi^-)$$

is again a neutrosophic bidirected graph.


 FIGURE 7. The neutrosophic bidirected graph $\mathcal{B}_N$.

 FIGURE 8. The orientation-reversed neutrosophic bidirected graph $\mathcal{B}_N^-$.

4. APPLICATIONS OF NEUTROSOPHIC BIDIRECTED GRAPHS

The theory of neutrosophic bidirected graphs provides a generalized and flexible mathematical framework for studying complex systems involving simultaneous uncertainty, indeterminacy, inconsistency, and directional interactions. Since the proposed structure extends fuzzy, intuitionistic fuzzy, and classical bidirected graph models, it possesses significant potential in both theoretical investigations and real-world applications.

Let $G = (V, E, \sigma, \mu, \eta, \phi, \beta)$ be a neutrosophic bidirected graph, where $\sigma = (T_\sigma, I_\sigma, F_\sigma)$ and $\mu = (T_\mu, I_\mu, F_\mu)$ denote the neutrosophic vertex and edge functions, respectively, and $\beta : E \rightarrow \{(-, -), (-, +), (+, -), (+, +)\}$ is the bidirection mapping.

The following applications illustrate the usefulness of this model.

4.1. Communication Networks. Consider a communication network in which each vertex represents a communication device and each edge represents a communication channel. For an edge $uv \in E$, $T_\mu(uv)$ may denote the reliability of successful transmission, $I_\mu(uv)$ the uncertainty caused by signal fluctuation, and $F_\mu(uv)$ the probability of communication failure.

The bidirection

$$\beta(uv) = (+, -)$$

indicates that the transmission from u to v is outgoing at u and incoming at v . Hence, neutrosophic bidirected graphs can model uncertain bidirectional communication systems more effectively than ordinary directed graphs.

4.2. Transportation Systems. Let vertices represent cities and edges represent roads connecting them. Suppose $uv \in E$. Then:

$$T_\mu(uv) = \text{degree of road availability,}$$

$$I_\mu(uv) = \text{uncertainty due to weather or traffic,}$$

and

$$F_\mu(uv) = \text{degree of blockage or failure.}$$

If $\beta(uv) = (+, +)$, then, the road allows transportation in both directions. Thus, neutrosophic bidirected graphs provide a mathematical tool for studying transportation systems under uncertain conditions.

4.3. Social Networks. In social networks, vertices correspond to individuals and edges represent relationships between them. For $uv \in E$, $T_\mu(uv)$ may represent the strength of friendship, $I_\mu(uv)$ the hesitation or ambiguity in the relationship, and $F_\mu(uv)$ the degree of conflict.

The bidirection function distinguishes mutual and non-mutual interactions. Therefore, the model is applicable in trust analysis, opinion dynamics, and recommendation systems.

4.4. Decision-Making Problems. Suppose $A = \{a_1, a_2, \dots, a_n\}$ is a collection of alternatives in a decision-making problem. A neutrosophic bidirected graph may be constructed by assigning each alternative to a vertex and defining edges according to preference relations.

For an edge $a_i a_j$, $T_\mu(a_i a_j)$ denotes the degree to which a_i is preferred over a_j , $I_\mu(a_i a_j)$ represents indeterminate assessment, and $F_\mu(a_i a_j)$ indicates rejection or opposition.

Hence, neutrosophic bidirected graphs are useful in multicriteria decision-making under uncertainty.

4.5. Biological Networks. In biological systems, vertices may represent neurons, proteins, or genes, while edges describe interactions among them. For example, $T_\mu(uv)$ can measure activation strength, $I_\mu(uv)$ the uncertainty in biological interaction, and $F_\mu(uv)$ the inhibitory effect.

The bidirection mapping models mutual biological interactions. Consequently, neutrosophic bidirected graphs are applicable in bioinformatics and neural network analysis.

4.6. Electrical and Power Networks. Let vertices denote substations and edges denote transmission lines. For an edge uv , $T_\mu(uv)$ represents operational efficiency, $I_\mu(uv)$ represents uncertain load variation, and $F_\mu(uv)$ represents the degree of line failure.

The bidirection mapping characterizes two-way power flow in smart grids. Therefore, neutrosophic bidirected graphs can be used in fault diagnosis and power distribution analysis.

To illustrate the proposed concept of neutrosophic bidirected graphs and the role of neutrosophic membership values together with bidirections, we present the following example. The example demonstrates how truth, indeterminacy, and falsity memberships can be assigned to vertices and edges, while bidirections describe the directional behavior of the connections in an uncertain communication network.

Example 4.1. Consider the neutrosophic bidirected graph $\mathcal{G} = (\mathcal{V}, \mathbf{E}, \sigma, \mu, \beta)$, where $V = \{\partial_1, \partial_2, \partial_3\}$ and $\mathbf{E} = \{\partial_1 \partial_2, \partial_2 \partial_3, \partial_1 \partial_3\}$. Define the neutrosophic vertex membership functions by

$$\sigma(\partial_1) = (0.8, 0.1, 0.2), \quad \sigma(\partial_2) = (0.7, 0.2, 0.3), \quad \text{and} \quad \sigma(\partial_3) = (0.9, 0.1, 0.1).$$

Further, define the neutrosophic edge membership functions as

$$\mu(\partial_1 \partial_2) = (0.6, 0.2, 0.3), \quad \mu(\partial_2 \partial_3) = (0.5, 0.3, 0.4), \quad \text{and} \quad \mu(\partial_1 \partial_3) = (0.7, 0.1, 0.2).$$

The bidirection mapping $\beta : \mathbf{E} \rightarrow \{(+, +), (+, -), (-, +), (-, -)\}$ is defined by

$$\beta(\partial_1 \partial_2) = (+, -), \quad \beta(\partial_2 \partial_3) = (-, +), \quad \text{and} \quad \beta(\partial_1 \partial_3) = (+, +).$$

This graph may represent a communication network in which:

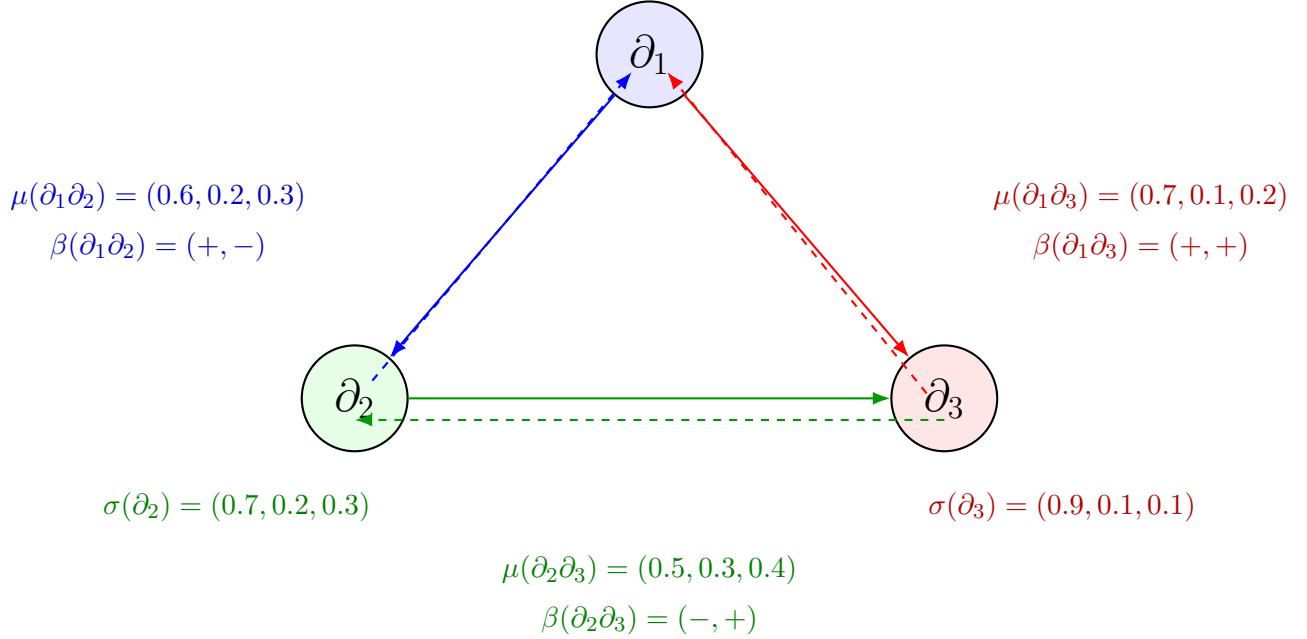
- (1) vertices correspond to communication devices,
- (2) edges represent communication links,
- (3) the first component of each neutrosophic value denotes the degree of reliable communication,
- (4) the second component denotes indeterminacy caused by unstable signals, and
- (5) the third component denotes the degree of transmission failure.

For example, $\mu(\partial_1 \partial_2) = (0.6, 0.2, 0.3)$ indicates that the communication link between ∂_1 and ∂_2 has reliability degree 0.6, indeterminacy degree 0.2, and failure degree 0.3. Moreover, $\beta(\partial_1 \partial_2) = (+, -)$ shows that the edge behaves as outgoing at ∂_1 and incoming at ∂_2 , while $\beta(\partial_1 \partial_3) = (+, +)$ represents a bidirectional interaction at both incident vertices.

Neutrosophic Bidirected Graph

$$\mathcal{G} = (\mathcal{V}, E, \sigma, \mu, \beta)$$

$$\sigma(\partial_1) = (0.8, 0.1, 0.2)$$



Legend (Bidirection β)	Information
$(+, +)$: bidirectional at both ends	• Vertices represent communication devices
$(+, -)$: outgoing at first, incoming at second	• Edges represent communication links
$(-, +)$: incoming at first, outgoing at second	• $(\top, \text{I}, \text{F})$ = truth, indeterminacy, falsity

FIGURE 9. Neutrosophic bidirected graph associated with Example 1.

Hence, G is a neutrosophic bidirected graph modeling an uncertain and partially indeterminate communication system.

5. CONCLUSION

In this paper, we introduced the concept of neutrosophic bidirected graphs as a natural extension of intuitionistic fuzzy bidirected graphs by incorporating truth-membership, indeterminacy-membership, and falsity-membership functions on vertices and edges together with a bidirection mapping. Several fundamental properties and structural characteristics of neutrosophic bidirected graphs were established.

We investigated reductions of neutrosophic bidirected graphs to fuzzy bidirected graphs, crisp bidirected graphs, and neutrosophic directed graphs under suitable conditions. Furthermore, operations such as intersection and union were studied, and important concepts including strong neutrosophic bidirected graphs, homomorphisms, and isomorphisms were analyzed in detail. The obtained results demonstrate that neutrosophic bidirected graphs provide a flexible and generalized framework for modeling uncertainty, indeterminacy, and inconsistency in bidirectional network structures.

The proposed structure may serve as a basis for future investigations in neutrosophic graph theory. Possible future directions include the study of neutrosophic connectivity, regularity, coloring, domination, spectral properties, and applications to optimization, decision-making, artificial intelligence, and complex network analysis.

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Part II — Neutrosophic Algebra**NEUTROSOPHIC LIE SUBALGEBRAS AND IDEALS: IMAGES, INVERSE IMAGES, AND QUOTIENT STRUCTURES**MUKHTAR AHMAD¹⁾ AND FLORENTIN SMARANDACHE^{2,*}¹⁾Algebra Research Group, Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung, Jalan Ganesha no. 10, Bandung, 40132, Indonesia²⁾University of New Mexico, Mathematics, Physics and Natural Sciences Division, Gallup, NM 87301, USA

*corresponding authors email: smarand@unm.edu

ABSTRACT. In this paper, we study neutrosophic Lie subalgebras and neutrosophic Lie ideals under Lie algebra homomorphisms. We investigate the behavior of these structures through images and inverse images and establish several preservation properties related to Lie algebra morphisms. We also examine sums of neutrosophic Lie ideals, upper level subsets, and quotient structures associated with kernels of homomorphisms. This work extends known results to the context of neutrosophic Lie algebras, contributing new insights into their behavior under algebraic mappings.

1. INTRODUCTION

The theory of uncertainty has become one of the most active and influential areas of modern mathematics due to its wide range of applications in algebra, engineering, artificial intelligence, information sciences, and decision-making problems. The pioneering work of [1] introduced the notion of fuzzy sets, which provided a mathematical framework for dealing with vagueness and imprecise information. Since then, fuzzy set theory has been extensively generalized and applied in many branches of mathematics and theoretical sciences.

To overcome certain limitations of fuzzy sets, [2] introduced intuitionistic fuzzy sets by incorporating both membership and non-membership functions. This extension provided a richer and more flexible framework for modeling uncertainty. Later, several algebraic structures based on intuitionistic fuzzy theory were developed, including intuitionistic fuzzy groups, rings, ideals, and Lie algebras [3, 4, 5]. These developments demonstrated the effectiveness of intuitionistic fuzzy methods in studying algebraic systems with uncertain behavior.

In another direction, [6] introduced complex fuzzy sets by extending membership grades from the real unit interval to the complex unit disc. This approach enabled the representation of periodic and phase-dependent uncertain information. Subsequently, [7] proposed the concept of complex intuitionistic fuzzy sets, which combines intuitionistic fuzzy theory with complex-valued membership structures. Since then, complex intuitionistic fuzzy algebraic structures have received considerable attention due to their applications in signal processing, pattern recognition, and intelligent systems [8, 9].

Neutrosophic set theory, initiated by [10], generalized fuzzy and intuitionistic fuzzy theories by introducing three independent components: truth-membership, indeterminacy-membership, and falsity-membership functions. Unlike intuitionistic fuzzy sets, neutrosophic sets allow the explicit modeling of indeterminate and inconsistent information. This additional degree of

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mukhku0786@gmail.com¹; 30125701@mahasiswa.itb.ac.id¹; ORCID: 0009-0006-4597-2278
smarand@unm.edu²; ORCID: 0000-0002-5560-5926.

freedom makes neutrosophic theory highly suitable for dealing with incomplete and contradictory data. Later, [11] introduced single-valued neutrosophic sets, which simplified the practical implementation of neutrosophic concepts and accelerated their applications in several mathematical disciplines.

In recent years, neutrosophic algebraic structures have attracted significant research interest. Various studies on neutrosophic groups, rings, modules, semigroups, and BCK/BCI-algebras have been carried out by many researchers [12, 13, 14, 15, 16, 17, 18]. The incorporation of neutrosophic ideas into algebraic systems has led to deeper insights into the role of uncertainty and indeterminacy in algebraic operations and homomorphic structures.

Lie algebras constitute one of the central topics in modern algebra due to their profound connections with geometry, topology, differential equations, and theoretical physics. They play a fundamental role in quantum mechanics, particle physics, symmetry theory, and representation theory [19, 20]. Because of their importance, fuzzy and intuitionistic fuzzy extensions of Lie algebras have been studied extensively over the last few decades [21, 22, 23]. These investigations focused mainly on fuzzy Lie subalgebras, fuzzy Lie ideals, intuitionistic fuzzy Lie structures, and their homomorphic properties.

The study of homomorphisms in fuzzy and neutrosophic algebraic structures is particularly important because homomorphisms preserve essential algebraic information between structures. Several authors have investigated images, inverse images, quotient structures, and level subsets in fuzzy and intuitionistic fuzzy algebraic systems [24, 25, 26]. However, the theory of neutrosophic Lie ideals and neutrosophic Lie subalgebras under Lie algebra homomorphisms remains comparatively underdeveloped.

Motivated by these observations, the present paper investigates neutrosophic Lie subalgebras and neutrosophic Lie ideals through Lie algebra homomorphisms. We study the behavior of neutrosophic Lie structures under images and inverse images and establish several preservation theorems. We further investigate sums of neutrosophic Lie ideals, upper level subsets, strong upper level subsets, and quotient structures associated with kernels of Lie algebra homomorphisms. In addition, we obtain correspondence-type results connecting quotient Lie algebras with neutrosophic Lie ideals.

The results established in this work generalize and extend many existing results from fuzzy Lie algebras, intuitionistic fuzzy Lie algebras, and complex intuitionistic fuzzy Lie algebras to the neutrosophic framework. Hence, this paper contributes to the growing interaction between neutrosophic mathematics and Lie algebra theory and opens new directions for further investigations in neutrosophic algebraic structures.

2. FUNDAMENTALS OF NEUTROSOPHIC LIE IDEALS

A neutrosophic set defined on a non-empty set X is denoted by $\mathbf{E} = (\mathbf{T}_{\mathbf{E}}, \mathbf{I}_{\mathbf{E}}, \mathbf{F}_{\mathbf{E}})$, where $\mathbf{T}_{\mathbf{E}}, \mathbf{I}_{\mathbf{E}}, \mathbf{F}_{\mathbf{E}} : X \rightarrow [0, 1]$ represent the truth-membership, indeterminacy-membership, and falsity-membership functions, respectively. For every $\mathbf{x} \in X$, the functions satisfy $0 \leq \mathbf{T}_{\mathbf{E}}(\mathbf{x}) + \mathbf{I}_{\mathbf{E}}(\mathbf{x}) + \mathbf{F}_{\mathbf{E}}(\mathbf{x}) \leq 3$. The concept of neutrosophic sets generalizes fuzzy sets and intuitionistic fuzzy sets. In particular:

- (1) if $\mathbf{I}_{\mathbf{E}}(\mathbf{x}) = 0$, then $\mathbf{E}$ reduces to an intuitionistic fuzzy set;
- (2) if additionally $\mathbf{F}_{\mathbf{E}}(\mathbf{x}) = 1 - \mathbf{T}_{\mathbf{E}}(\mathbf{x})$, then $\mathbf{E}$ becomes a classical fuzzy set.

Let Ξ be a Lie algebra over a field $\mathbb{F}$. A neutrosophic set $\mathbf{E} = (\mathbf{T}_{\mathbf{E}}, \mathbf{I}_{\mathbf{E}}, \mathbf{F}_{\mathbf{E}})$ on Ξ is called a neutrosophic Lie ideal of Ξ if, for all $\mathbf{x}, \beta \in \Xi$ and $\alpha \in \mathbb{F}$, the following conditions hold:

- (1) $\mathbf{T}_{\mathbf{E}}(\mathbf{x} + \beta) \geq \min\{\mathbf{T}_{\mathbf{E}}(\mathbf{x}), \mathbf{T}_{\mathbf{E}}(\beta)\}$,
- (2) $\mathbf{I}_{\mathbf{E}}(\mathbf{x} + \beta) \geq \min\{\mathbf{I}_{\mathbf{E}}(\mathbf{x}), \mathbf{I}_{\mathbf{E}}(\beta)\}$,
- (3) $\mathbf{F}_{\mathbf{E}}(\mathbf{x} + \beta) \leq \max\{\mathbf{F}_{\mathbf{E}}(\mathbf{x}), \mathbf{F}_{\mathbf{E}}(\beta)\}$,
- (4) $\mathbf{T}_{\mathbf{E}}(\alpha\mathbf{x}) \geq \mathbf{T}_{\mathbf{E}}(\mathbf{x})$,
- (5) $\mathbf{I}_{\mathbf{E}}(\alpha\mathbf{x}) \geq \mathbf{I}_{\mathbf{E}}(\mathbf{x})$,

- (6) $\mathbf{F}_{\mathbf{E}}(\alpha \mathbf{x}) \leq \mathbf{F}_{\mathbf{E}}(\mathbf{x})$,
- (7) $\mathbf{T}_{\mathbf{E}}([\mathbf{x}, \beta]) \geq \min\{\mathbf{T}_{\mathbf{E}}(\mathbf{x}), \mathbf{T}_{\mathbf{E}}(\beta)\}$,
- (8) $\mathbf{I}_{\mathbf{E}}([\mathbf{x}, \beta]) \geq \min\{\mathbf{I}_{\mathbf{E}}(\mathbf{x}), \mathbf{I}_{\mathbf{E}}(\beta)\}$,
- (9) $\mathbf{F}_{\mathbf{E}}([\mathbf{x}, \beta]) \leq \max\{\mathbf{F}_{\mathbf{E}}(\mathbf{x}), \mathbf{F}_{\mathbf{E}}(\beta)\}$.

Definition 2.1. Let $\mathbf{E} = (\mathbf{T}_{\mathbf{E}}, \mathbf{I}_{\mathbf{E}}, \mathbf{F}_{\mathbf{E}})$ be a neutrosophic set defined on a Lie algebra Ξ over a field $\mathbb{F}$. Then, $\mathbf{E}$ is called a neutrosophic Lie subalgebra of Ξ if, for all $\mathbf{x}, \beta \in \Xi$ and $\alpha \in \mathbb{F}$, the following conditions hold:

$$(1) \quad \begin{aligned} \mathbf{T}_{\mathbf{E}}(\mathbf{x} + \beta) &\geq \mathbf{T}_{\mathbf{E}}(\mathbf{x}) \wedge \mathbf{T}_{\mathbf{E}}(\beta), \\ \mathbf{I}_{\mathbf{E}}(\mathbf{x} + \beta) &\geq \mathbf{I}_{\mathbf{E}}(\mathbf{x}) \wedge \mathbf{I}_{\mathbf{E}}(\beta), \end{aligned}$$

and

$$\mathbf{F}_{\mathbf{E}}(\mathbf{x} + \beta) \leq \mathbf{F}_{\mathbf{E}}(\mathbf{x}) \vee \mathbf{F}_{\mathbf{E}}(\beta);$$

$$(2) \quad \begin{aligned} \mathbf{T}_{\mathbf{E}}(\alpha \mathbf{x}) &\geq \mathbf{T}_{\mathbf{E}}(\mathbf{x}), \\ \mathbf{I}_{\mathbf{E}}(\alpha \mathbf{x}) &\geq \mathbf{I}_{\mathbf{E}}(\mathbf{x}), \end{aligned}$$

and

$$\mathbf{F}_{\mathbf{E}}(\alpha \mathbf{x}) \leq \mathbf{F}_{\mathbf{E}}(\mathbf{x});$$

$$(3) \quad \begin{aligned} \mathbf{T}_{\mathbf{E}}([\mathbf{x}, \beta]) &\geq \mathbf{T}_{\mathbf{E}}(\mathbf{x}) \wedge \mathbf{T}_{\mathbf{E}}(\beta), \\ \mathbf{I}_{\mathbf{E}}([\mathbf{x}, \beta]) &\geq \mathbf{I}_{\mathbf{E}}(\mathbf{x}) \wedge \mathbf{I}_{\mathbf{E}}(\beta), \end{aligned}$$

and

$$\mathbf{F}_{\mathbf{E}}([\mathbf{x}, \beta]) \leq \mathbf{F}_{\mathbf{E}}(\mathbf{x}) \vee \mathbf{F}_{\mathbf{E}}(\beta).$$

If condition (3) is replaced by

$$\begin{aligned} \mathbf{T}_{\mathbf{E}}([\mathbf{x}, \beta]) &\geq \mathbf{T}_{\mathbf{E}}(\mathbf{x}) \vee \mathbf{T}_{\mathbf{E}}(\beta), \\ \mathbf{I}_{\mathbf{E}}([\mathbf{x}, \beta]) &\geq \mathbf{I}_{\mathbf{E}}(\mathbf{x}) \vee \mathbf{I}_{\mathbf{E}}(\beta), \end{aligned}$$

and

$$\mathbf{F}_{\mathbf{E}}([\mathbf{x}, \beta]) \leq \mathbf{F}_{\mathbf{E}}(\mathbf{x}) \wedge \mathbf{F}_{\mathbf{E}}(\beta).$$

Then, $\mathbf{E}$ is called a neutrosophic Lie ideal of Ξ .

3. MAPPINGS AND INVERSE MAPPINGS OF NEUTROSOPHIC LIE SUBALGEBRAS (IDEALS) THROUGH LIE HOMOMORPHISMS

Let Ξ_1 and Ξ_2 be Lie algebras, let $\mathbf{E} = (\mathbf{T}_{\mathbf{E}}, \mathbf{I}_{\mathbf{E}}, \mathbf{F}_{\mathbf{E}})$ be a neutrosophic subset of Ξ_1 , and let $\aleph : \Xi_1 \rightarrow \Xi_2$ be a mapping. The neutrosophic subset $\aleph(\mathbf{E}) = (\mathbf{T}_{\aleph(\mathbf{E})}, \mathbf{I}_{\aleph(\mathbf{E})}, \mathbf{F}_{\aleph(\mathbf{E})})$ of $\aleph(\Xi_1)$ is defined by, for every $\beta \in \aleph(\Xi_1)$,

$$\mathbf{T}_{\aleph(\mathbf{E})}(\beta) = \sup_{\mathbf{x} \in \aleph^{-1}(\{\beta\})} \{\mathbf{T}_{\mathbf{E}}(\mathbf{x})\},$$

$$\mathbf{I}_{\aleph(\mathbf{E})}(\beta) = \sup_{\mathbf{x} \in \aleph^{-1}(\{\beta\})} \{\mathbf{I}_{\mathbf{E}}(\mathbf{x})\},$$

and

$$\mathbf{F}_{\aleph(\mathbf{E})}(\beta) = \sup_{\mathbf{x} \in \aleph^{-1}(\{\beta\})} \{\mathbf{F}_{\mathbf{E}}(\mathbf{x})\}.$$

The neutrosophic subset $\aleph(\mathbf{E})$ is called the image (mapping) of $\mathbf{E}$ under $\aleph$.

Similarly, let $\mathbf{P} = (\mathbf{T}_{\mathbf{P}}, \mathbf{I}_{\mathbf{P}}, \mathbf{F}_{\mathbf{P}})$ be a neutrosophic subset of Ξ_2 . The inverse image of $\mathbf{P}$ under $\aleph$, denoted by $\aleph^{-1}(\mathbf{P})$, is defined as $\aleph^{-1}(\mathbf{P}) = (\mathbf{T}_{\aleph^{-1}(\mathbf{P})}, \mathbf{I}_{\aleph^{-1}(\mathbf{P})}, \mathbf{F}_{\aleph^{-1}(\mathbf{P})})$, where for every $\mathbf{x} \in \Xi_1$,

$$\mathbf{T}_{\aleph^{-1}(\mathbf{P})}(\mathbf{x}) = \mathbf{T}_{\mathbf{P}}(\aleph(\mathbf{x})),$$

$$\mathbf{I}_{\aleph^{-1}(\mathbf{P})}(\mathbf{x}) = \mathbf{I}_{\mathbf{P}}(\aleph(\mathbf{x})),$$

and

$$\mathbf{F}_{\aleph^{-1}(\mathbf{P})}(\mathbf{x}) = \mathbf{F}_{\mathbf{P}}(\aleph(\mathbf{x})).$$

Theorem 3.1. *Let $\aleph : \Xi_1 \rightarrow \Xi_2$ be a Lie algebra homomorphism. If $\mathbf{P} = (\mathbf{T}_{\mathbf{P}}, \mathbf{I}_{\mathbf{P}}, \mathbf{F}_{\mathbf{P}})$ is a neutrosophic Lie subalgebra of Ξ_2 , then the inverse image $\aleph^{-1}(\mathbf{P})$ is a neutrosophic Lie subalgebra of Ξ_1 .*

Proof. To prove that $\aleph^{-1}(\mathbf{P}) = (\mathbf{T}_{\aleph^{-1}(\mathbf{P})}, \mathbf{I}_{\aleph^{-1}(\mathbf{P})}, \mathbf{F}_{\aleph^{-1}(\mathbf{P})})$ is a neutrosophic Lie subalgebra of Ξ_1 , we verify the defining conditions. Let $\mathbf{x}, \beta \in \Xi_1$ and $\alpha \in \mathbb{F}$. Since

$$\mathbf{T}_{\aleph^{-1}(\mathbf{P})}(\mathbf{x}) = \mathbf{T}_{\mathbf{P}}(\aleph(\mathbf{x})), \mathbf{I}_{\aleph^{-1}(\mathbf{P})}(\mathbf{x}) = \mathbf{I}_{\mathbf{P}}(\aleph(\mathbf{x})), \text{ and } \mathbf{F}_{\aleph^{-1}(\mathbf{P})}(\mathbf{x}) = \mathbf{F}_{\mathbf{P}}(\aleph(\mathbf{x})),$$

we proceed as follows.

(i) **Addition:** Since $\aleph$ is linear,

$$\aleph(\mathbf{x} + \beta) = \aleph(\mathbf{x}) + \aleph(\beta).$$

Therefore,

$$\mathbf{T}_{\aleph^{-1}(\mathbf{P})}(\mathbf{x} + \beta) = \mathbf{T}_{\mathbf{P}}(\aleph(\mathbf{x} + \beta)) = \mathbf{T}_{\mathbf{P}}(\aleph(\mathbf{x}) + \aleph(\beta)).$$

Because $\mathbf{P}$ is a neutrosophic Lie subalgebra of Ξ_2 ,

$$\mathbf{T}_{\mathbf{P}}(\aleph(\mathbf{x}) + \aleph(\beta)) \geq \mathbf{T}_{\mathbf{P}}(\aleph(\mathbf{x})) \wedge \mathbf{T}_{\mathbf{P}}(\aleph(\beta)).$$

Hence,

$$\mathbf{T}_{\aleph^{-1}(\mathbf{P})}(\mathbf{x} + \beta) \geq \mathbf{T}_{\aleph^{-1}(\mathbf{P})}(\mathbf{x}) \wedge \mathbf{T}_{\aleph^{-1}(\mathbf{P})}(\beta).$$

Similarly,

$$\begin{aligned} \mathbf{I}_{\aleph^{-1}(\mathbf{P})}(\mathbf{x} + \beta) &= \mathbf{I}_{\mathbf{P}}(\aleph(\mathbf{x} + \beta)) = \mathbf{I}_{\mathbf{P}}(\aleph(\mathbf{x}) + \aleph(\beta)) \\ &\geq \mathbf{I}_{\mathbf{P}}(\aleph(\mathbf{x})) \wedge \mathbf{I}_{\mathbf{P}}(\aleph(\beta)), \end{aligned}$$

so

$$\mathbf{I}_{\aleph^{-1}(\mathbf{P})}(\mathbf{x} + \beta) \geq \mathbf{I}_{\aleph^{-1}(\mathbf{P})}(\mathbf{x}) \wedge \mathbf{I}_{\aleph^{-1}(\mathbf{P})}(\beta).$$

Also,

$$\begin{aligned} \mathbf{F}_{\aleph^{-1}(\mathbf{P})}(\mathbf{x} + \beta) &= \mathbf{F}_{\mathbf{P}}(\aleph(\mathbf{x} + \beta)) = \mathbf{F}_{\mathbf{P}}(\aleph(\mathbf{x}) + \aleph(\beta)) \\ &\leq \mathbf{F}_{\mathbf{P}}(\aleph(\mathbf{x})) \vee \mathbf{F}_{\mathbf{P}}(\aleph(\beta)), \end{aligned}$$

and therefore

$$\mathbf{F}_{\aleph^{-1}(\mathbf{P})}(\mathbf{x} + \beta) \leq \mathbf{F}_{\aleph^{-1}(\mathbf{P})}(\mathbf{x}) \vee \mathbf{F}_{\aleph^{-1}(\mathbf{P})}(\beta).$$

(ii) **Scalar Multiplication:** For any $\alpha \in \mathbb{F}$,

$$\aleph(\alpha\mathbf{x}) = \alpha\aleph(\mathbf{x}).$$

Hence,

$$\mathbf{T}_{\aleph^{-1}(\mathbf{P})}(\alpha\mathbf{x}) = \mathbf{T}_{\mathbf{P}}(\aleph(\alpha\mathbf{x})) = \mathbf{T}_{\mathbf{P}}(\alpha\aleph(\mathbf{x})) \geq \mathbf{T}_{\mathbf{P}}(\aleph(\mathbf{x})),$$

which implies

$$\mathbf{T}_{\aleph^{-1}(\mathbf{P})}(\alpha\mathbf{x}) \geq \mathbf{T}_{\aleph^{-1}(\mathbf{P})}(\mathbf{x}).$$

Similarly,

$$\mathbf{I}_{\aleph^{-1}(\mathbf{P})}(\alpha\mathbf{x}) = \mathbf{I}_{\mathbf{P}}(\alpha\aleph(\mathbf{x})) \geq \mathbf{I}_{\mathbf{P}}(\aleph(\mathbf{x})).$$

Thus,

$$\mathbf{I}_{\aleph^{-1}(\mathbf{P})}(\alpha\mathbf{x}) \geq \mathbf{I}_{\aleph^{-1}(\mathbf{P})}(\mathbf{x}).$$

Also,

$$\mathbf{F}_{\aleph^{-1}(\mathbf{P})}(\alpha\mathbf{x}) = \mathbf{F}_{\mathbf{P}}(\alpha\aleph(\mathbf{x})) \leq \mathbf{F}_{\mathbf{P}}(\aleph(\mathbf{x})).$$

So,

$$\mathbf{F}_{\aleph^{-1}(\mathbf{P})}(\alpha\mathbf{x}) \leq \mathbf{F}_{\aleph^{-1}(\mathbf{P})}(\mathbf{x}).$$

(iii) **Lie Bracket:** Since $\aleph$ is a Lie algebra homomorphism,

$$\aleph([\mathbf{x}, \beta]) = [\aleph(\mathbf{x}), \aleph(\beta)].$$

Therefore,

$$\mathbf{T}_{\aleph^{-1}(\mathbf{P})}([\mathbf{x}, \beta]) = \mathbf{T}_{\mathbf{P}}(\aleph([\mathbf{x}, \beta])) = \mathbf{T}_{\mathbf{P}}([\aleph(\mathbf{x}), \aleph(\beta)]).$$

Since $\mathbf{P}$ is a neutrosophic Lie subalgebra,

$$\mathbf{T}_{\mathbf{P}}([\aleph(\mathbf{x}), \aleph(\beta)]) \geq \mathbf{T}_{\mathbf{P}}(\aleph(\mathbf{x})) \wedge \mathbf{T}_{\mathbf{P}}(\aleph(\beta)).$$

Hence,

$$\mathbf{T}_{\aleph^{-1}(\mathbf{P})}([\mathbf{x}, \beta]) \geq \mathbf{T}_{\aleph^{-1}(\mathbf{P})}(\mathbf{x}) \wedge \mathbf{T}_{\aleph^{-1}(\mathbf{P})}(\beta).$$

Similarly,

$$\mathbf{I}_{\aleph^{-1}(\mathbf{P})}([\mathbf{x}, \beta]) = \mathbf{I}_{\mathbf{P}}([\aleph(\mathbf{x}), \aleph(\beta)]) \geq \mathbf{I}_{\mathbf{P}}(\aleph(\mathbf{x})) \wedge \mathbf{I}_{\mathbf{P}}(\aleph(\beta)).$$

Thus,

$$\mathbf{I}_{\aleph^{-1}(\mathbf{P})}([\mathbf{x}, \beta]) \geq \mathbf{I}_{\aleph^{-1}(\mathbf{P})}(\mathbf{x}) \wedge \mathbf{I}_{\aleph^{-1}(\mathbf{P})}(\beta).$$

Also,

$$\mathbf{F}_{\aleph^{-1}(\mathbf{P})}([\mathbf{x}, \beta]) = \mathbf{F}_{\mathbf{P}}([\aleph(\mathbf{x}), \aleph(\beta)]) \leq \mathbf{F}_{\mathbf{P}}(\aleph(\mathbf{x})) \vee \mathbf{F}_{\mathbf{P}}(\aleph(\beta)),$$

which implies

$$\mathbf{F}_{\aleph^{-1}(\mathbf{P})}([\mathbf{x}, \beta]) \leq \mathbf{F}_{\aleph^{-1}(\mathbf{P})}(\mathbf{x}) \vee \mathbf{F}_{\aleph^{-1}(\mathbf{P})}(\beta).$$

Therefore, $\aleph^{-1}(\mathbf{P})$ satisfies all the conditions of a neutrosophic Lie subalgebra of Ξ_1 . $\square$

Corollary 3.1. *Let $\aleph : \Xi_1 \rightarrow \Xi_2$ be a Lie algebra homomorphism. If $\mathbf{P} = (\mathbf{T}_{\mathbf{P}}, \mathbf{I}_{\mathbf{P}}, \mathbf{F}_{\mathbf{P}})$ is a neutrosophic Lie ideal of Ξ_2 . Then, the inverse image $\aleph^{-1}(\mathbf{P})$ is also a neutrosophic Lie ideal of Ξ_1 .*

Proof. The proof follows similarly to Theorem 1. We only verify the Lie ideal condition involving the Lie bracket.

Let $\mathbf{x}, \beta \in \Xi_1$. Since $\aleph$ is a Lie algebra homomorphism,

$$\aleph([\mathbf{x}, \beta]) = [\aleph(\mathbf{x}), \aleph(\beta)].$$

For the truth-membership function, we have

$$\mathbf{T}_{\aleph^{-1}(\mathbf{P})}([\mathbf{x}, \beta]) = \mathbf{T}_{\mathbf{P}}(\aleph([\mathbf{x}, \beta])) = \mathbf{T}_{\mathbf{P}}([\aleph(\mathbf{x}), \aleph(\beta)]).$$

Since $\mathbf{P}$ is a neutrosophic Lie ideal of Ξ_2 ,

$$\mathbf{T}_{\mathbf{P}}([\aleph(\mathbf{x}), \aleph(\beta)]) \geq \mathbf{T}_{\mathbf{P}}(\aleph(\mathbf{x})) \vee \mathbf{T}_{\mathbf{P}}(\aleph(\beta)).$$

Hence,

$$\mathbf{T}_{\aleph^{-1}(\mathbf{P})}([\mathbf{x}, \beta]) \geq \mathbf{T}_{\aleph^{-1}(\mathbf{P})}(\mathbf{x}) \vee \mathbf{T}_{\aleph^{-1}(\mathbf{P})}(\beta).$$

Similarly, for the indeterminacy-membership function,

$$\mathbf{I}_{\aleph^{-1}(\mathbf{P})}([\mathbf{x}, \beta]) = \mathbf{I}_{\mathbf{P}}(\aleph([\mathbf{x}, \beta])) = \mathbf{I}_{\mathbf{P}}([\aleph(\mathbf{x}), \aleph(\beta)]) \geq \mathbf{I}_{\mathbf{P}}(\aleph(\mathbf{x})) \wedge \mathbf{I}_{\mathbf{P}}(\aleph(\beta)),$$

and therefore

$$\mathbf{I}_{\aleph^{-1}(\mathbf{P})}([\mathbf{x}, \beta]) \geq \mathbf{I}_{\aleph^{-1}(\mathbf{P})}(\mathbf{x}) \wedge \mathbf{I}_{\aleph^{-1}(\mathbf{P})}(\beta).$$

For the falsity-membership function,

$$\mathbf{F}_{\aleph^{-1}(\mathbf{P})}([\mathbf{x}, \beta]) = \mathbf{F}_{\mathbf{P}}(\aleph([\mathbf{x}, \beta])) = \mathbf{F}_{\mathbf{P}}([\aleph(\mathbf{x}), \aleph(\beta)]).$$

Since $\mathbf{P}$ is a neutrosophic Lie ideal,

$$\mathbf{F}_{\mathbf{P}}([\aleph(\mathbf{x}), \aleph(\beta)]) \leq \mathbf{F}_{\mathbf{P}}(\aleph(\mathbf{x})) \wedge \mathbf{F}_{\mathbf{P}}(\aleph(\beta)).$$

Thus,

$$\mathbf{F}_{\aleph^{-1}(\mathbf{P})}([\mathbf{x}, \beta]) \leq \mathbf{F}_{\aleph^{-1}(\mathbf{P})}(\mathbf{x}) \wedge \mathbf{F}_{\aleph^{-1}(\mathbf{P})}(\beta).$$

Therefore, $\aleph^{-1}(\mathbf{P})$ satisfies all the conditions of a neutrosophic Lie ideal of Ξ_1 . $\square$

Theorem 3.2. *Let $\aleph : \Xi_1 \rightarrow \Xi_2$ be a Lie algebra homomorphism. If $\mathbf{E} = (\mathbf{T}_{\mathbf{E}}, \mathbf{I}_{\mathbf{E}}, \mathbf{F}_{\mathbf{E}})$ is a neutrosophic Lie subalgebra of Ξ_1 . Then, the image $\aleph(\mathbf{E})$ is a neutrosophic Lie subalgebra of $\aleph(\Xi_1)$.*

Proof. First, we show that $\aleph(\mathbf{E}) = (\mathbf{T}_{\aleph(\mathbf{E})}, \mathbf{I}_{\aleph(\mathbf{E})}, \mathbf{F}_{\aleph(\mathbf{E})})$ is a neutrosophic set on $\aleph(\Xi_1)$. For every $\beta \in \aleph(\Xi_1)$,

$$\mathbf{T}_{\aleph(\mathbf{E})}(\beta) = \sup_{\beta = \aleph(\mathbf{x})} \{\mathbf{T}_{\mathbf{E}}(\mathbf{x})\}, \quad \mathbf{I}_{\aleph(\mathbf{E})}(\beta) = \sup_{\beta = \aleph(\mathbf{x})} \{\mathbf{I}_{\mathbf{E}}(\mathbf{x})\}, \quad \text{and} \quad \mathbf{F}_{\aleph(\mathbf{E})}(\beta) = \sup_{\beta = \aleph(\mathbf{x})} \{\mathbf{F}_{\mathbf{E}}(\mathbf{x})\}.$$

Next, we verify the defining conditions of a neutrosophic Lie subalgebra.

Let $\beta_1, \beta_2 \in \aleph(\Xi_1)$. Then there exist $\mathbf{x}_1, \mathbf{x}_2 \in \Xi_1$ such that

$$\aleph(\mathbf{x}_1) = \beta_1 \quad \text{and} \quad \aleph(\mathbf{x}_2) = \beta_2.$$

(i) Since $\aleph$ is linear,

$$\beta_1 + \beta_2 = \aleph(\mathbf{x}_1) + \aleph(\mathbf{x}_2) = \aleph(\mathbf{x}_1 + \mathbf{x}_2).$$

Therefore,

$$\begin{aligned} \mathbf{T}_{\aleph(\mathbf{E})}(\beta_1 + \beta_2) &= \sup_{\beta_1 + \beta_2 = \aleph(z)} \{\mathbf{T}_{\mathbf{E}}(z)\} \\ &\geq \mathbf{T}_{\mathbf{E}}(\mathbf{x}_1 + \mathbf{x}_2). \end{aligned}$$

Since $\mathbf{E}$ is a neutrosophic Lie subalgebra of Ξ_1 ,

$$\mathbf{T}_{\mathbf{E}}(\mathbf{x}_1 + \mathbf{x}_2) \geq \mathbf{T}_{\mathbf{E}}(\mathbf{x}_1) \wedge \mathbf{T}_{\mathbf{E}}(\mathbf{x}_2).$$

Hence,

$$\mathbf{T}_{\aleph(\mathbf{E})}(\beta_1 + \beta_2) \geq \mathbf{T}_{\aleph(\mathbf{E})}(\beta_1) \wedge \mathbf{T}_{\aleph(\mathbf{E})}(\beta_2).$$

Similarly,

$$\mathbf{I}_{\aleph(\mathbf{E})}(\beta_1 + \beta_2) \geq \mathbf{I}_{\aleph(\mathbf{E})}(\beta_1) \wedge \mathbf{I}_{\aleph(\mathbf{E})}(\beta_2),$$

and

$$\mathbf{F}_{\aleph(\mathbf{E})}(\beta_1 + \beta_2) \leq \mathbf{F}_{\aleph(\mathbf{E})}(\beta_1) \vee \mathbf{F}_{\aleph(\mathbf{E})}(\beta_2).$$

(ii) Let $\alpha \in \mathbb{F}$. Since

$$\alpha\beta_1 = \alpha\aleph(\mathbf{x}_1) = \aleph(\alpha\mathbf{x}_1),$$

we obtain

$$\mathbf{T}_{\aleph(\mathbf{E})}(\alpha\beta_1) = \sup_{\alpha\beta_1 = \aleph(z)} \{\mathbf{T}_{\mathbf{E}}(z)\} \geq \mathbf{T}_{\mathbf{E}}(\alpha\mathbf{x}_1).$$

Since $\mathbf{E}$ is a neutrosophic Lie subalgebra,

$$\mathbf{T}_{\mathbf{E}}(\alpha\mathbf{x}_1) \geq \mathbf{T}_{\mathbf{E}}(\mathbf{x}_1).$$

Thus,

$$\mathbf{T}_{\aleph(\mathbf{E})}(\alpha\beta_1) \geq \mathbf{T}_{\aleph(\mathbf{E})}(\beta_1).$$

Similarly,

$$\mathbf{I}_{\aleph(\mathbf{E})}(\alpha\beta_1) \geq \mathbf{I}_{\aleph(\mathbf{E})}(\beta_1),$$

and

$$\mathbf{F}_{\aleph(\mathbf{E})}(\alpha\beta_1) \leq \mathbf{F}_{\aleph(\mathbf{E})}(\beta_1).$$

(iii) Since $\aleph$ is a Lie algebra homomorphism,

$$[\beta_1, \beta_2] = [\aleph(\mathbf{x}_1), \aleph(\mathbf{x}_2)] = \aleph([\mathbf{x}_1, \mathbf{x}_2]).$$

Therefore,

$$\mathbf{T}_{\aleph(\mathbf{E})}([\beta_1, \beta_2]) = \sup_{[\beta_1, \beta_2] = \aleph(z)} \{\mathbf{T}_{\mathbf{E}}(z)\} \geq \mathbf{T}_{\mathbf{E}}([\mathbf{x}_1, \mathbf{x}_2]).$$

Because $\mathbf{E}$ is a neutrosophic Lie subalgebra,

$$\mathbf{T}_{\mathbf{E}}([\mathbf{x}_1, \mathbf{x}_2]) \geq \mathbf{T}_{\mathbf{E}}(\mathbf{x}_1) \wedge \mathbf{T}_{\mathbf{E}}(\mathbf{x}_2).$$

Hence,

$$\mathbf{T}_{\aleph(\mathbf{E})}([\beta_1, \beta_2]) \geq \mathbf{T}_{\aleph(\mathbf{E})}(\beta_1) \wedge \mathbf{T}_{\aleph(\mathbf{E})}(\beta_2).$$

Similarly,

$$\mathbf{I}_{\aleph(\mathbf{E})}([\beta_1, \beta_2]) \geq \mathbf{I}_{\aleph(\mathbf{E})}(\beta_1) \wedge \mathbf{I}_{\aleph(\mathbf{E})}(\beta_2),$$

and

$$\mathbf{F}_{\aleph(\mathbf{E})}([\beta_1, \beta_2]) \leq \mathbf{F}_{\aleph(\mathbf{E})}(\beta_1) \vee \mathbf{F}_{\aleph(\mathbf{E})}(\beta_2).$$

Therefore, $\aleph(\mathbf{E})$ is a neutrosophic Lie subalgebra of $\aleph(\Xi_1)$. $\square$

Corollary 3.2. *Let $\aleph : \Xi_1 \rightarrow \Xi_2$ be a Lie algebra homomorphism. If $\mathbf{E} = (\mathbf{T}_{\mathbf{E}}, \mathbf{I}_{\mathbf{E}}, \mathbf{F}_{\mathbf{E}})$ is a neutrosophic Lie ideal of Ξ_1 , then the image $\aleph(\mathbf{E})$ is a neutrosophic Lie ideal of $\text{im}(\aleph)$.*

Proof. It is sufficient to verify the Lie ideal conditions for the Lie bracket.

Let $\beta_1, \beta_2 \in \text{im}(\aleph)$. Then, there exist $\mathbf{x}_1, \mathbf{x}_2 \in \Xi_1$ such that

$$\aleph(\mathbf{x}_1) = \beta_1 \quad \text{and} \quad \aleph(\mathbf{x}_2) = \beta_2.$$

Since $\aleph$ is a Lie algebra homomorphism, we have

$$[\beta_1, \beta_2] = [\aleph(\mathbf{x}_1), \aleph(\mathbf{x}_2)] = \aleph([\mathbf{x}_1, \mathbf{x}_2]),$$

and,

$$\mathbf{T}_{\aleph(\mathbf{E})}([\beta_1, \beta_2]) = \sup_{[\beta_1, \beta_2] = \aleph(z)} \{\mathbf{T}_{\mathbf{E}}(z)\}.$$

In particular,

$$\mathbf{T}_{\aleph(\mathbf{E})}([\beta_1, \beta_2]) \geq \mathbf{T}_{\mathbf{E}}([\mathbf{x}_1, \mathbf{x}_2]).$$

Since $\mathbf{E}$ is a neutrosophic Lie ideal of Ξ_1 ,

$$\mathbf{T}_{\mathbf{E}}([\mathbf{x}_1, \mathbf{x}_2]) \geq \mathbf{T}_{\mathbf{E}}(\mathbf{x}_1) \vee \mathbf{T}_{\mathbf{E}}(\mathbf{x}_2).$$

Hence,

$$\mathbf{T}_{\aleph(\mathbf{E})}([\beta_1, \beta_2]) \geq \mathbf{T}_{\aleph(\mathbf{E})}(\beta_1) \vee \mathbf{T}_{\aleph(\mathbf{E})}(\beta_2).$$

Similarly,

$$\mathbf{I}_{\aleph(\mathbf{E})}([\beta_1, \beta_2]) = \sup_{[\beta_1, \beta_2] = \aleph(z)} \{\mathbf{I}_{\mathbf{E}}(z)\} \geq \mathbf{I}_{\mathbf{E}}([\mathbf{x}_1, \mathbf{x}_2]).$$

Since $\mathbf{E}$ is a neutrosophic Lie ideal,

$$\mathbf{I}_{\mathbf{E}}([\mathbf{x}_1, \mathbf{x}_2]) \geq \mathbf{I}_{\mathbf{E}}(\mathbf{x}_1) \wedge \mathbf{I}_{\mathbf{E}}(\mathbf{x}_2).$$

Therefore,

$$\mathbf{I}_{\aleph(\mathbf{E})}([\beta_1, \beta_2]) \geq \mathbf{I}_{\aleph(\mathbf{E})}(\beta_1) \wedge \mathbf{I}_{\aleph(\mathbf{E})}(\beta_2).$$

For the falsity-membership function,

$$\mathbf{F}_{\aleph(\mathbf{E})}([\beta_1, \beta_2]) = \in \mathbf{F}_{[\beta_1, \beta_2] = \aleph(z)} \{\mathbf{F}_{\mathbf{E}}(z)\}.$$

Hence,

$$\mathbf{F}_{\aleph(\mathbf{E})}([\beta_1, \beta_2]) \leq \mathbf{F}_{\mathbf{E}}([\mathbf{x}_1, \mathbf{x}_2]).$$

Since $\mathbf{E}$ is a neutrosophic Lie ideal,

$$\mathbf{F}_{\mathbf{E}}([\mathbf{x}_1, \mathbf{x}_2]) \leq \mathbf{F}_{\mathbf{E}}(\mathbf{x}_1) \wedge \mathbf{F}_{\mathbf{E}}(\mathbf{x}_2).$$

Thus,

$$\mathbf{F}_{\aleph(\mathbf{E})}([\beta_1, \beta_2]) \leq \mathbf{F}_{\aleph(\mathbf{E})}(\beta_1) \wedge \mathbf{F}_{\aleph(\mathbf{E})}(\beta_2).$$

Therefore, $\aleph(\mathbf{E})$ satisfies all the conditions of a neutrosophic Lie ideal of $\text{im}(\aleph)$. $\square$

Corollary 3.3. *Let $\aleph : \Xi_1 \rightarrow \Xi_2$ be a Lie algebra homomorphism. If $\mathbf{E} = (\mathbf{T}_{\mathbf{E}}, \mathbf{I}_{\mathbf{E}}, \mathbf{F}_{\mathbf{E}})$ is a neutrosophic Lie ideal of Ξ_1 , then the image $\aleph(\mathbf{E})$ is a neutrosophic Lie ideal of $\text{im}(\aleph)$.*

Proof. Let $\beta_1, \beta_2 \in \text{im}(\aleph)$ and let $\alpha \in \mathbb{F}$. Then, there exist $\mathbf{x}_1, \mathbf{x}_2 \in \Xi_1$ such that $\aleph(\mathbf{x}_1) = \beta_1$ and $\aleph(\mathbf{x}_2) = \beta_2$.

Now, we verify the defining conditions of a neutrosophic Lie ideal.

(i) Closure under addition:

Since $\aleph$ is linear,

$$\beta_1 + \beta_2 = \aleph(\mathbf{x}_1) + \aleph(\mathbf{x}_2) = \aleph(\mathbf{x}_1 + \mathbf{x}_2).$$

Hence,

$$\mathbf{T}_{\aleph(\mathbf{E})}(\beta_1 + \beta_2) = \sup_{\beta_1 + \beta_2 = \aleph(z)} \{\mathbf{T}_{\mathbf{E}}(z)\} \geq \mathbf{T}_{\mathbf{E}}(\mathbf{x}_1 + \mathbf{x}_2).$$

Because $\mathbf{E}$ is a neutrosophic Lie ideal of Ξ_1 ,

$$\mathbf{T}_{\mathbf{E}}(\mathbf{x}_1 + \mathbf{x}_2) \geq \mathbf{T}_{\mathbf{E}}(\mathbf{x}_1) \wedge \mathbf{T}_{\mathbf{E}}(\mathbf{x}_2).$$

Therefore,

$$\mathbf{T}_{\aleph(\mathbf{E})}(\beta_1 + \beta_2) \geq \mathbf{T}_{\aleph(\mathbf{E})}(\beta_1) \wedge \mathbf{T}_{\aleph(\mathbf{E})}(\beta_2).$$

Similarly,

$$\begin{aligned} \mathbf{I}_{\aleph(\mathbf{E})}(\beta_1 + \beta_2) &= \sup_{\beta_1 + \beta_2 = \aleph(z)} \{\mathbf{I}_{\mathbf{E}}(z)\} \geq \mathbf{I}_{\mathbf{E}}(\mathbf{x}_1 + \mathbf{x}_2) \\ &\geq \mathbf{I}_{\mathbf{E}}(\mathbf{x}_1) \wedge \mathbf{I}_{\mathbf{E}}(\mathbf{x}_2), \end{aligned}$$

which gives

$$\mathbf{I}_{\aleph(\mathbf{E})}(\beta_1 + \beta_2) \geq \mathbf{I}_{\aleph(\mathbf{E})}(\beta_1) \wedge \mathbf{I}_{\aleph(\mathbf{E})}(\beta_2).$$

Also,

$$\begin{aligned} \mathbf{F}_{\aleph(\mathbf{E})}(\beta_1 + \beta_2) &= \mathbf{F}_{\beta_1 + \beta_2 = \aleph(z)} \{\mathbf{F}_{\mathbf{E}}(z)\} \leq \mathbf{F}_{\mathbf{E}}(\mathbf{x}_1 + \mathbf{x}_2) \\ &\leq \mathbf{F}_{\mathbf{E}}(\mathbf{x}_1) \vee \mathbf{F}_{\mathbf{E}}(\mathbf{x}_2). \end{aligned}$$

Thus,

$$\mathbf{F}_{\aleph(\mathbf{E})}(\beta_1 + \beta_2) \leq \mathbf{F}_{\aleph(\mathbf{E})}(\beta_1) \vee \mathbf{F}_{\aleph(\mathbf{E})}(\beta_2).$$

(ii) Scalar multiplication: Since

$$\alpha\beta_1 = \alpha\aleph(\mathbf{x}_1) = \aleph(\alpha\mathbf{x}_1),$$

we have

$$\mathbf{T}_{\aleph(\mathbf{E})}(\alpha\beta_1) = \sup_{\alpha\beta_1 = \aleph(z)} \{\mathbf{T}_{\mathbf{E}}(z)\} \geq \mathbf{T}_{\mathbf{E}}(\alpha\mathbf{x}_1).$$

Since $\mathbf{E}$ is a neutrosophic Lie ideal,

$$\mathbf{T}_{\mathbf{E}}(\alpha\mathbf{x}_1) \geq \mathbf{T}_{\mathbf{E}}(\mathbf{x}_1).$$

Hence,

$$\mathbf{T}_{\aleph(\mathbf{E})}(\alpha\beta_1) \geq \mathbf{T}_{\aleph(\mathbf{E})}(\beta_1).$$

Similarly,

$$\mathbf{I}_{\aleph(\mathbf{E})}(\alpha\beta_1) \geq \mathbf{I}_{\aleph(\mathbf{E})}(\beta_1),$$

and

$$\mathbf{F}_{\aleph(\mathbf{E})}(\alpha\beta_1) \leq \mathbf{F}_{\aleph(\mathbf{E})}(\beta_1).$$

(iii) Lie bracket condition: Since $\aleph$ is a Lie algebra homomorphism,

$$[\beta_1, \beta_2] = [\aleph(\mathbf{x}_1), \aleph(\mathbf{x}_2)] = \aleph([\mathbf{x}_1, \mathbf{x}_2]).$$

Therefore,

$$\mathbf{T}_{\aleph(\mathbf{E})}([\beta_1, \beta_2]) = \sup_{[\beta_1, \beta_2] = \aleph(z)} \{\mathbf{T}_{\mathbf{E}}(z)\} \geq \mathbf{T}_{\mathbf{E}}([\mathbf{x}_1, \mathbf{x}_2]).$$

Because $\mathbf{E}$ is a neutrosophic Lie ideal,

$$\mathbf{T}_{\mathbf{E}}([\mathbf{x}_1, \mathbf{x}_2]) \geq \mathbf{T}_{\mathbf{E}}(\mathbf{x}_1) \vee \mathbf{T}_{\mathbf{E}}(\mathbf{x}_2).$$

Thus,

$$\mathbf{T}_{\aleph(\mathbf{E})}([\beta_1, \beta_2]) \geq \mathbf{T}_{\aleph(\mathbf{E})}(\beta_1) \vee \mathbf{T}_{\aleph(\mathbf{E})}(\beta_2).$$

Similarly,

$$\begin{aligned} \mathbf{I}_{\aleph(\mathbf{E})}([\beta_1, \beta_2]) &= \sup_{[\beta_1, \beta_2] = \aleph(z)} \{\mathbf{I}_{\mathbf{E}}(z)\} \geq \mathbf{I}_{\mathbf{E}}([\mathbf{x}_1, \mathbf{x}_2]) \\ &\geq \mathbf{I}_{\mathbf{E}}(\mathbf{x}_1) \wedge \mathbf{I}_{\mathbf{E}}(\mathbf{x}_2). \end{aligned}$$

Hence,

$$\mathbf{I}_{\aleph(\mathbf{E})}([\beta_1, \beta_2]) \geq \mathbf{I}_{\aleph(\mathbf{E})}(\beta_1) \wedge \mathbf{I}_{\aleph(\mathbf{E})}(\beta_2).$$

Also,

$$\mathbf{F}_{\aleph(\mathbf{E})}([\beta_1, \beta_2]) = \mathbf{F}_{[\beta_1, \beta_2] = \aleph(z)} \{\mathbf{F}_{\mathbf{E}}(z)\} \leq \mathbf{F}_{\mathbf{E}}([\mathbf{x}_1, \mathbf{x}_2]).$$

Since $\mathbf{E}$ is a neutrosophic Lie ideal,

$$\mathbf{F}_{\mathbf{E}}([\mathbf{x}_1, \mathbf{x}_2]) \leq \mathbf{F}_{\mathbf{E}}(\mathbf{x}_1) \wedge \mathbf{F}_{\mathbf{E}}(\mathbf{x}_2).$$

Therefore,

$$\mathbf{F}_{\aleph(\mathbf{E})}([\beta_1, \beta_2]) \leq \mathbf{F}_{\aleph(\mathbf{E})}(\beta_1) \wedge \mathbf{F}_{\aleph(\mathbf{E})}(\beta_2).$$

Consequently, $\aleph(\mathbf{E})$ satisfies all the defining conditions of a neutrosophic Lie ideal of $\text{im}(\aleph)$. $\square$

Theorem 3.3. *Let $\aleph : \Xi_1 \rightarrow \Xi_2$ be an injective Lie algebra homomorphism and let $\mathbf{E} = (\mathbf{T}_{\mathbf{E}}, \mathbf{I}_{\mathbf{E}}, \mathbf{F}_{\mathbf{E}})$ be a neutrosophic Lie subalgebra of Ξ_1 . Then $\aleph^{-1}(\aleph(\mathbf{E})) = \mathbf{E}$.*

Proof. Let $\mathbf{E}^* = \aleph^{-1}(\aleph(\mathbf{E}))$. We show that the truth-membership, indeterminacy-membership, and falsity-membership functions of $\mathbf{E}^*$ coincide with those of $\mathbf{E}$.

Let $x \in \Xi_1$ be arbitrary. Since $\aleph$ is injective,

$$\aleph^{-1}(\{\aleph(\mathbf{x})\}) = \{\mathbf{x}\}.$$

Therefore,

$$\mathbf{T}_{\aleph(\mathbf{E})}(\aleph(\mathbf{x})) = \sup_{\mathbf{y} \in \aleph^{-1}(\{\aleph(\mathbf{x})\})} \mathbf{T}_{\mathbf{E}}(\mathbf{y}) = \mathbf{T}_{\mathbf{E}}(\mathbf{x}).$$

Hence,

$$\mathbf{T}_{\mathbf{E}^*}(\mathbf{x}) = \mathbf{T}_{\aleph(\mathbf{E})}(\aleph(\mathbf{x})) = \mathbf{T}_{\mathbf{E}}(\mathbf{x}).$$

Similarly,

$$\mathbf{I}_{\mathbf{E}^*}(\mathbf{x}) = \mathbf{I}_{\aleph(\mathbf{E})}(\aleph(\mathbf{x})) = \mathbf{I}_{\mathbf{E}}(\mathbf{x}).$$

Also,

$$\mathbf{F}_{\aleph(\mathbf{E})}(\aleph(\mathbf{x})) = \inf_{\mathbf{y} \in \aleph^{-1}(\{\aleph(\mathbf{x})\})} \mathbf{F}_{\mathbf{E}}(\mathbf{y}) = \mathbf{F}_{\mathbf{E}}(\mathbf{x}),$$

and therefore,

$$\mathbf{F}_{\mathbf{E}^*}(\mathbf{x}) = \mathbf{F}_{\mathbf{E}}(\mathbf{x}).$$

Thus,

$$\mathbf{T}_{\mathbf{E}^*} = \mathbf{T}_{\mathbf{E}}, \quad \mathbf{I}_{\mathbf{E}^*} = \mathbf{I}_{\mathbf{E}}, \quad \mathbf{F}_{\mathbf{E}^*} = \mathbf{F}_{\mathbf{E}}.$$

Hence,

$$\mathbf{E}^* = \mathbf{E}.$$

Therefore,

$$\aleph^{-1}(\aleph(\mathbf{E})) = \mathbf{E}.$$

$\square$

Theorem 3.4. *Let $\aleph : \Xi_1 \rightarrow \Xi_2$ be a surjective Lie algebra homomorphism and let $\mathbf{P} = (\mathbf{T}_{\mathbf{P}}, \mathbf{I}_{\mathbf{P}}, \mathbf{F}_{\mathbf{P}})$ be a neutrosophic Lie subalgebra of Ξ_2 . Then $\aleph(\aleph^{-1}(\mathbf{P})) = \mathbf{P}$.*

Proof. Let $P^* = \aleph(\aleph^{-1}(P))$. Let $\beta \in \Xi_2$. Since $\aleph$ is surjective,

$$\aleph^{-1}(\{\beta\}) \neq \emptyset.$$

For every $x \in \aleph^{-1}(\{\beta\})$, we have

$$\mathbf{T}_{\aleph^{-1}(P)}(\mathbf{x}) = \mathbf{T}_P(\aleph(\mathbf{x})) = \mathbf{T}_P(\beta).$$

Consequently,

$$\mathbf{T}_{P^*}(\beta) = \sup_{x \in \aleph^{-1}(\{\beta\})} \mathbf{T}_{\aleph^{-1}(P)}(\mathbf{x}) = \mathbf{T}_P(\beta).$$

Similarly,

$$\mathbf{I}_{P^*}(\beta) = \sup_{x \in \aleph^{-1}(\{\beta\})} \mathbf{I}_{\aleph^{-1}(P)}(\mathbf{x}) = \mathbf{I}_P(\beta).$$

Also,

$$\mathbf{F}_{P^*}(\beta) = \inf_{x \in \aleph^{-1}(\{\beta\})} \mathbf{F}_{\aleph^{-1}(P)}(\mathbf{x}) = \mathbf{F}_P(\beta).$$

Therefore,

$$\mathbf{T}_{P^*} = \mathbf{T}_P, \quad \mathbf{I}_{P^*} = \mathbf{I}_P, \quad \mathbf{F}_{P^*} = \mathbf{F}_P.$$

Hence,

$$P^* = P.$$

Thus,

$$\aleph(\aleph^{-1}(P)) = P.$$

□

Theorem 3.5. Let $\aleph : \Xi_1 \rightarrow \Xi_2$ be a Lie algebra homomorphism and let $\mathbf{E}$ be a neutrosophic subset of Ξ_1 . Then $\mathbf{E} \subseteq \aleph^{-1}(\aleph(\mathbf{E}))$.

Proof. Let $\mathbf{E}^* = \aleph^{-1}(\aleph(\mathbf{E}))$. Take any $x \in \Xi_1$. Since

$$\mathbf{x} \in \aleph^{-1}(\{\aleph(\mathbf{x})\}),$$

it follows that

$$\mathbf{T}_{\aleph(\mathbf{E})}(\aleph(\mathbf{x})) = \sup_{\mathbf{y} \in \aleph^{-1}(\{\aleph(\mathbf{x})\})} \mathbf{T}_{\mathbf{E}}(\mathbf{y}) \geq \mathbf{T}_{\mathbf{E}}(\mathbf{x}).$$

Therefore,

$$\mathbf{T}_{\mathbf{E}^*}(\mathbf{x}) \geq \mathbf{T}_{\mathbf{E}}(\mathbf{x}).$$

Similarly,

$$\mathbf{I}_{\mathbf{E}^*}(\mathbf{x}) \geq \mathbf{I}_{\mathbf{E}}(\mathbf{x}).$$

Furthermore,

$$\mathbf{F}_{\aleph(\mathbf{E})}(\aleph(\mathbf{x})) = \inf_{\mathbf{y} \in \aleph^{-1}(\{\aleph(\mathbf{x})\})} \mathbf{F}_{\mathbf{E}}(\mathbf{y}) \leq \mathbf{F}_{\mathbf{E}}(\mathbf{x}),$$

which implies

$$\mathbf{F}_{\mathbf{E}^*}(\mathbf{x}) \leq \mathbf{F}_{\mathbf{E}}(\mathbf{x}).$$

Hence,

$$\mathbf{T}_{\mathbf{E}} \leq \mathbf{T}_{\mathbf{E}^*}, \quad \mathbf{I}_{\mathbf{E}} \leq \mathbf{I}_{\mathbf{E}^*}, \quad \mathbf{F}_{\mathbf{E}} \geq \mathbf{F}_{\mathbf{E}^*}.$$

Therefore,

$$\mathbf{E} \subseteq \mathbf{E}^*.$$

Thus,

$$\mathbf{E} \subseteq \aleph^{-1}(\aleph(\mathbf{E})).$$

□

Theorem 3.6. Let $\aleph : \Xi_1 \rightarrow \Xi_2$ be a Lie algebra homomorphism and let P be a neutrosophic subset of Ξ_2 . Then $\aleph(\aleph^{-1}(P)) \subseteq P$.

Proof. Let $P^* = \aleph(\aleph^{-1}(P))$. Take any $\beta \in \text{im}(\aleph)$. Now, we choose $x \in \Xi_1$ such that $\aleph(x) = \beta$. For every $y \in \aleph^{-1}(\{\beta\})$, we have

$$\mathbf{T}_{\aleph^{-1}(P)}(y) = \mathbf{T}_P(\aleph(y)) = \mathbf{T}_P(\beta).$$

Hence,

$$\mathbf{T}_{P^*}(\beta) = \sup_{y \in \aleph^{-1}(\{\beta\})} \mathbf{T}_{\aleph^{-1}(P)}(y) = \mathbf{T}_P(\beta).$$

Similarly,

$$\mathbf{I}_{P^*}(\beta) = \mathbf{I}_P(\beta),$$

and

$$\mathbf{F}_{P^*}(\beta) = \mathbf{F}_P(\beta).$$

Therefore,

$$P^* = P \Big|_{\text{im}(\aleph)}.$$

Consequently,

$$\aleph(\aleph^{-1}(P)) = P^* \subseteq P.$$

Hence,

$$\aleph(\aleph^{-1}(P)) \subseteq P.$$

□

Theorem 3.7. *Let $\aleph : \Xi_1 \rightarrow \Xi_2$ be a Lie algebra isomorphism. Then there is a one-to-one correspondence between the set of all neutrosophic Lie subalgebras of Ξ_1 and the set of all neutrosophic Lie subalgebras of Ξ_2 given by $\mathbf{E} \mapsto \aleph(\mathbf{E})$.*

Proof. Define

$$\Phi : \mathcal{NS}(\Xi_1) \longrightarrow \mathcal{NS}(\Xi_2)$$

by

$$\Phi(\mathbf{E}) = \aleph(\mathbf{E}),$$

where $\mathcal{NS}(\Xi_i)$ denotes the collection of all neutrosophic Lie subalgebras of Ξ_i .

Since $\aleph$ is a Lie algebra homomorphism, the image of every neutrosophic Lie subalgebra of Ξ_1 is a neutrosophic Lie subalgebra of Ξ_2 . Hence Φ is well-defined.

We first show that Φ is injective. Suppose

$$\Phi(\mathbf{E}_1) = \Phi(\mathbf{E}_2).$$

Then

$$\aleph(\mathbf{E}_1) = \aleph(\mathbf{E}_2).$$

Applying $\aleph^{-1}$ to both sides yields

$$\aleph^{-1}(\aleph(\mathbf{E}_1)) = \aleph^{-1}(\aleph(\mathbf{E}_2)).$$

Since $\aleph$ is injective,

$$\aleph^{-1}(\aleph(\mathbf{E}_i)) = \mathbf{E}_i \quad (i = 1, 2).$$

Therefore

$$\mathbf{E}_1 = \mathbf{E}_2.$$

Hence Φ is injective.

Next we show that Φ is surjective. Let P be any neutrosophic Lie subalgebra of Ξ_2 . Since $\aleph$ is a Lie algebra homomorphism,

$$\aleph^{-1}(P)$$

is a neutrosophic Lie subalgebra of Ξ_1 .

Because $\aleph$ is surjective,

$$\aleph(\aleph^{-1}(P)) = P.$$

Thus

$$\Phi(\aleph^{-1}(\mathbf{P})) = \mathbf{P}.$$

Hence Φ is surjective.

Therefore Φ is bijective, establishing a one-to-one correspondence between neutrosophic Lie subalgebras of Ξ_1 and neutrosophic Lie subalgebras of Ξ_2 . $\square$

Corollary 3.4. *Let $\aleph : \Xi_1 \rightarrow \Xi_2$ be a Lie algebra isomorphism. Then there is a one-to-one correspondence between the neutrosophic Lie ideals of Ξ_1 and the neutrosophic Lie ideals of Ξ_2 .*

Theorem 3.8. *Let $\aleph : \Xi_1 \rightarrow \Xi_2$ be a Lie algebra homomorphism and let $\{\mathbf{P}_\lambda\}_{\lambda \in \Lambda}$ be a family of neutrosophic Lie subalgebras of Ξ_2 . Then $\aleph^{-1}\left(\bigcap_{\lambda \in \Lambda} \mathbf{P}_\lambda\right) = \bigcap_{\lambda \in \Lambda} \aleph^{-1}(\mathbf{P}_\lambda)$.*

Proof. Let $\mathbf{P} = \bigcap_{\lambda \in \Lambda} \mathbf{P}_\lambda$. For every $x \in \Xi_1$,

$$\mathbf{T}_{\mathbf{P}}(\aleph(\mathbf{x})) = \inf_{\lambda \in \Lambda} \mathbf{T}_{\mathbf{P}_\lambda}(\aleph(\mathbf{x})).$$

Therefore,

$$\mathbf{T}_{\aleph^{-1}(\mathbf{P})}(\mathbf{x}) = \inf_{\lambda \in \Lambda} \mathbf{T}_{\aleph^{-1}(\mathbf{P}_\lambda)}(\mathbf{x}).$$

Similarly,

$$\mathbf{I}_{\aleph^{-1}(\mathbf{P})}(\mathbf{x}) = \inf_{\lambda \in \Lambda} \mathbf{I}_{\aleph^{-1}(\mathbf{P}_\lambda)}(\mathbf{x}),$$

and

$$\mathbf{F}_{\aleph^{-1}(\mathbf{P})}(\mathbf{x}) = \sup_{\lambda \in \Lambda} \mathbf{F}_{\aleph^{-1}(\mathbf{P}_\lambda)}(\mathbf{x}).$$

These are precisely the membership functions of

$$\bigcap_{\lambda \in \Lambda} \aleph^{-1}(\mathbf{P}_\lambda).$$

Hence,

$$\aleph^{-1}\left(\bigcap_{\lambda \in \Lambda} \mathbf{P}_\lambda\right) = \bigcap_{\lambda \in \Lambda} \aleph^{-1}(\mathbf{P}_\lambda).$$

$\square$

Theorem 3.9. *Let $\aleph : \Xi_1 \rightarrow \Xi_2$ be a surjective Lie algebra homomorphism. If $\mathbf{E}$ is a neutrosophic Lie ideal of Ξ_1 such that $\ker(\aleph) \subseteq \mathbf{E}$, then $\aleph^{-1}(\aleph(\mathbf{E})) = \mathbf{E}$.*

Proof. In view of Theorem 3.5, we have

$$\mathbf{E} \subseteq \aleph^{-1}(\aleph(\mathbf{E})).$$

It remains to prove the reverse inclusion. Let $\mathbf{x} \in \aleph^{-1}(\aleph(\mathbf{E}))$. Then, there exists $e \in \mathbf{E}$ such that

$$\aleph(\mathbf{x}) = \aleph(e).$$

Hence,

$$\aleph(\mathbf{x} - e) = 0,$$

so that

$$\mathbf{x} - e \in \ker(\aleph).$$

Since

$$\ker(\aleph) \subseteq \mathbf{E},$$

we have

$$\mathbf{x} - e \in \mathbf{E}.$$

Because $\mathbf{E}$ is a neutrosophic Lie ideal and therefore a neutrosophic Lie subalgebra, it is closed under addition. Thus,

$$\mathbf{x} = (\mathbf{x} - e) + e \in \mathbf{E}.$$

Hence,

$$\aleph^{-1}(\aleph(\mathbf{E})) \subseteq \mathbf{E}.$$

Combining this with

$$\mathbf{E} \subseteq \aleph^{-1}(\aleph(\mathbf{E})),$$

we obtain

$$\aleph^{-1}(\aleph(\mathbf{E})) = \mathbf{E}.$$

□

Theorem 3.10. *Let $\aleph : \Xi_1 \rightarrow \Xi_2$ be a surjective Lie algebra homomorphism. Then, the mapping $\mathbf{E} \mapsto \aleph(\mathbf{E})$ establishes a bijection between*

- (1) *the neutrosophic Lie ideals of Ξ_1 containing $\ker(\aleph)$, and*
- (2) *the neutrosophic Lie ideals of Ξ_2 .*

The inverse correspondence is given by $\mathbf{P} \mapsto \aleph^{-1}(\mathbf{P})$.

Proof. Let

$$\mathcal{A} = \{\mathbf{E} : \mathbf{E} \text{ is a neutrosophic Lie ideal of } \Xi_1, \ker(\aleph) \subseteq \mathbf{E}\}$$

and

$$\mathcal{B} = \{\mathbf{P} : \mathbf{P} \text{ is a neutrosophic Lie ideal of } \Xi_2\}.$$

Define

$$\Phi : \mathcal{A} \rightarrow \mathcal{B}, \quad \Phi(\mathbf{E}) = \aleph(\mathbf{E}).$$

Since the image of a neutrosophic Lie ideal under a Lie homomorphism is again a neutrosophic Lie ideal, Φ is well-defined.

Define

$$\Psi : \mathcal{B} \rightarrow \mathcal{A}, \quad \Psi(\mathbf{P}) = \aleph^{-1}(\mathbf{P}).$$

Since inverse images of neutrosophic Lie ideals are neutrosophic Lie ideals, Ψ is well-defined.

Moreover, $\ker(\aleph) = \aleph^{-1}(\{0\}) \subseteq \aleph^{-1}(\mathbf{P})$, because every ideal contains the zero element.

Thus, $\Psi(\mathbf{P}) \in \mathcal{A}$. Now, let $\mathbf{E} \in \mathcal{A}$. By Theorem 3.9,

$$\Psi(\Phi(\mathbf{E})) = \aleph^{-1}(\aleph(\mathbf{E})) = \mathbf{E}.$$

Likewise, for every $\mathbf{P} \in \mathcal{B}$,

$$\Phi(\Psi(\mathbf{P})) = \aleph(\aleph^{-1}(\mathbf{P})) = \mathbf{P},$$

since $\aleph$ is surjective. Therefore,

$$\Psi \circ \Phi = \text{id}_{\mathcal{A}}$$

and

$$\Phi \circ \Psi = \text{id}_{\mathcal{B}}.$$

Hence, Φ and Ψ are mutually inverse bijections.

Consequently, the correspondence $\mathbf{E} \longleftrightarrow \aleph(\mathbf{E})$ is a bijection between neutrosophic Lie ideals of Ξ_1 containing $\ker(\aleph)$ and neutrosophic Lie ideals of Ξ_2 . □

4. MORE HOMOMORPHISM PROPERTIES OF NEUTROSOPHIC LIE ALGEBRAS

Let $\mathbf{E} = (\mathbf{T}_{\mathbf{E}}, \mathbf{I}_{\mathbf{E}}, \mathbf{F}_{\mathbf{E}})$ and $\mathbf{P} = (\mathbf{T}_{\mathbf{P}}, \mathbf{I}_{\mathbf{P}}, \mathbf{F}_{\mathbf{P}})$ be two neutrosophic sets on a Lie algebra Ξ over a field $\mathbb{F}$. The sum of $\mathbf{E}$ and $\mathbf{P}$ is defined by $\mathbf{E} + \mathbf{P} = (\mathbf{T}_{\mathbf{E}+\mathbf{P}}, \mathbf{I}_{\mathbf{E}+\mathbf{P}}, \mathbf{F}_{\mathbf{E}+\mathbf{P}})$, where for every $\mathbf{x} \in \Xi$,

$$\mathbf{T}_{\mathbf{E}+\mathbf{P}}(\mathbf{x}) = \sup_{\mathbf{x}=\mathbf{x}_1+\mathbf{x}_2} \{\mathbf{T}_{\mathbf{E}}(\mathbf{x}_1) \wedge \mathbf{T}_{\mathbf{P}}(\mathbf{x}_2)\},$$

$$\mathbf{I}_{\mathbf{E}+\mathbf{P}}(\mathbf{x}) = \sup_{\mathbf{x}=\mathbf{x}_1+\mathbf{x}_2} \{\mathbf{I}_{\mathbf{E}}(\mathbf{x}_1) \wedge \mathbf{I}_{\mathbf{P}}(\mathbf{x}_2)\},$$

and

$$\mathbf{F}_{\mathbf{E}+\mathbf{P}}(\mathbf{x}) = \sup_{\mathbf{x}=\mathbf{x}_1+\mathbf{x}_2} \{\mathbf{F}_{\mathbf{E}}(\mathbf{x}_1) \vee \mathbf{F}_{\mathbf{P}}(\mathbf{x}_2)\}.$$

Now, let

$$\mathbf{E} = (\mathbf{T}_E, \mathbf{I}_E, \mathbf{F}_E) \quad \text{and} \quad Q = (\mathbf{T}_Q, \mathbf{I}_Q, \mathbf{F}_Q)$$

be two neutrosophic sets on the same set Ξ .

We say that $\mathbf{E}$ is homogeneous with Q if, for all $\mathbf{x}, \beta \in \Xi$, the following conditions hold:

- (1) $\mathbf{T}_E(\mathbf{x}) \leq \mathbf{T}_Q(\beta)$;
- (2) $\mathbf{I}_E(\mathbf{x}) \leq \mathbf{I}_Q(\beta)$;
- (3) $\mathbf{F}_E(\mathbf{x}) \leq \mathbf{F}_Q(\beta)$.

Theorem 4.1. *Let $\mathbf{E} = (\mathbf{T}_E, \mathbf{I}_E, \mathbf{F}_E)$ and $\mathbf{P} = (\mathbf{T}_P, \mathbf{I}_P, \mathbf{F}_P)$ be two neutrosophic Lie ideals of a Lie algebra Ξ over a field $\mathbb{F}$. Assume that $\mathbf{E}$ is homogeneous with $\mathbf{P}$ and that the sum $\mathbf{E} + \mathbf{P}$ is homogeneous. Then, $\mathbf{E} + \mathbf{P}$ is a neutrosophic Lie ideal of Ξ .*

Proof. Let

$$\hat{\mathbf{A}} = \mathbf{E} + \mathbf{P} = (\mathbf{T}_{\mathbf{E}+\mathbf{P}}, \mathbf{I}_{\mathbf{E}+\mathbf{P}}, \mathbf{F}_{\mathbf{E}+\mathbf{P}}).$$

Now, we show that $\hat{\mathbf{A}}$ satisfies all the defining conditions of a neutrosophic Lie ideal. Let $\mathbf{x}, \beta \in \Xi$ and $\alpha \in \mathbb{F}$. Then, we have,

$$\begin{aligned} \mathbf{T}_{\hat{\mathbf{A}}}(\mathbf{x}) &= \sup_{\mathbf{x}=\mathbf{x}_1+\mathbf{x}_2} \{\mathbf{T}_E(\mathbf{x}_1) \wedge \mathbf{T}_P(\mathbf{x}_2)\}, \\ \mathbf{I}_{\hat{\mathbf{A}}}(\mathbf{x}) &= \sup_{\mathbf{x}=\mathbf{x}_1+\mathbf{x}_2} \{\mathbf{I}_E(\mathbf{x}_1) \wedge \mathbf{I}_P(\mathbf{x}_2)\}, \end{aligned}$$

and

$$\mathbf{F}_{\hat{\mathbf{A}}}(\mathbf{x}) = \in \mathbf{F}_{\mathbf{x}=\mathbf{x}_1+\mathbf{x}_2} \{\mathbf{F}_E(\mathbf{x}_1) \vee \mathbf{F}_P(\mathbf{x}_2)\}.$$

(i) Closure under addition: Take decompositions

$$\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2 \quad \text{and} \quad \beta = \beta_1 + \beta_2,$$

where

$$\mathbf{x}_1, \beta_1 \in \Xi \quad \text{and} \quad \mathbf{x}_2, \beta_2 \in \Xi.$$

Then,

$$\mathbf{x} + \beta = (\mathbf{x}_1 + \beta_1) + (\mathbf{x}_2 + \beta_2).$$

Hence,

$$\begin{aligned} \mathbf{T}_{\hat{\mathbf{A}}}(\mathbf{x} + \beta) &= \sup_{\mathbf{x}+\beta=u+v} \{\mathbf{T}_E(u) \wedge \mathbf{T}_P(v)\} \\ &\geq \mathbf{T}_E(\mathbf{x}_1 + \beta_1) \wedge \mathbf{T}_P(\mathbf{x}_2 + \beta_2). \end{aligned}$$

Since $\mathbf{E}$ and $\mathbf{P}$ are neutrosophic Lie ideals,

$$\mathbf{T}_E(\mathbf{x}_1 + \beta_1) \geq \mathbf{T}_E(\mathbf{x}_1) \wedge \mathbf{T}_E(\beta_1),$$

and

$$\mathbf{T}_P(\mathbf{x}_2 + \beta_2) \geq \mathbf{T}_P(\mathbf{x}_2) \wedge \mathbf{T}_P(\beta_2).$$

Therefore,

$$\mathbf{T}_{\hat{\mathbf{A}}}(\mathbf{x} + \beta) \geq (\mathbf{T}_E(\mathbf{x}_1) \wedge \mathbf{T}_E(\beta_1)) \wedge (\mathbf{T}_P(\mathbf{x}_2) \wedge \mathbf{T}_P(\beta_2)).$$

Using associativity and commutativity of $\wedge$,

$$\mathbf{T}_{\hat{\mathbf{A}}}(\mathbf{x} + \beta) \geq (\mathbf{T}_E(\mathbf{x}_1) \wedge \mathbf{T}_P(\mathbf{x}_2)) \wedge (\mathbf{T}_E(\beta_1) \wedge \mathbf{T}_P(\beta_2)).$$

Taking supremum over all decompositions, we obtain

$$\mathbf{T}_{\hat{\mathbf{A}}}(\mathbf{x} + \beta) \geq \mathbf{T}_{\hat{\mathbf{A}}}(\mathbf{x}) \wedge \mathbf{T}_{\hat{\mathbf{A}}}(\beta).$$

Similarly,

$$\mathbf{I}_{\hat{\mathbf{A}}}(\mathbf{x} + \beta) \geq \mathbf{I}_{\hat{\mathbf{A}}}(\mathbf{x}) \wedge \mathbf{I}_{\hat{\mathbf{A}}}(\beta).$$

For the falsity-membership function,

$$\mathbf{F}_{\hat{\mathbf{A}}}(\mathbf{x} + \beta) = \in \mathbf{F}_{\mathbf{x}+\beta=u+v} \{\mathbf{F}_E(u) \vee \mathbf{F}_P(v)\}.$$

Since

$$\mathbf{F}_{\mathbf{E}}(\mathbf{x}_1 + \beta_1) \leq \mathbf{F}_{\mathbf{E}}(\mathbf{x}_1) \vee \mathbf{F}_{\mathbf{E}}(\beta_1),$$

and

$$\mathbf{F}_{\mathbf{P}}(\mathbf{x}_2 + \beta_2) \leq \mathbf{F}_{\mathbf{P}}(\mathbf{x}_2) \vee \mathbf{F}_{\mathbf{P}}(\beta_2),$$

we obtain

$$\mathbf{F}_{\hat{\mathbf{A}}}(\mathbf{x} + \beta) \leq \mathbf{F}_{\hat{\mathbf{A}}}(\mathbf{x}) \vee \mathbf{F}_{\hat{\mathbf{A}}}(\beta).$$

(ii) Scalar multiplication: Let $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$. Then, $\alpha\mathbf{x} = \alpha\mathbf{x}_1 + \alpha\mathbf{x}_2$. Hence,

$$\mathbf{T}_{\hat{\mathbf{A}}}(\alpha\mathbf{x}) \geq \mathbf{T}_{\mathbf{E}}(\alpha\mathbf{x}_1) \wedge \mathbf{T}_{\mathbf{P}}(\alpha\mathbf{x}_2).$$

Since $\mathbf{E}$ and $\mathbf{P}$ are neutrosophic Lie ideals,

$$\mathbf{T}_{\mathbf{E}}(\alpha\mathbf{x}_1) \geq \mathbf{T}_{\mathbf{E}}(\mathbf{x}_1),$$

and

$$\mathbf{T}_{\mathbf{P}}(\alpha\mathbf{x}_2) \geq \mathbf{T}_{\mathbf{P}}(\mathbf{x}_2).$$

Thus,

$$\mathbf{T}_{\hat{\mathbf{A}}}(\alpha\mathbf{x}) \geq \mathbf{T}_{\mathbf{E}}(\mathbf{x}_1) \wedge \mathbf{T}_{\mathbf{P}}(\mathbf{x}_2).$$

Taking supremum over all decompositions,

$$\mathbf{T}_{\hat{\mathbf{A}}}(\alpha\mathbf{x}) \geq \mathbf{T}_{\hat{\mathbf{A}}}(\mathbf{x}).$$

Similarly,

$$\mathbf{I}_{\hat{\mathbf{A}}}(\alpha\mathbf{x}) \geq \mathbf{I}_{\hat{\mathbf{A}}}(\mathbf{x}),$$

and

$$\mathbf{F}_{\hat{\mathbf{A}}}(\alpha\mathbf{x}) \leq \mathbf{F}_{\hat{\mathbf{A}}}(\mathbf{x}).$$

(iii) Lie bracket condition: Take decompositions

$$\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2 \quad \text{and} \quad \beta = \beta_1 + \beta_2.$$

Using bilinearity of the Lie bracket,

$$[\mathbf{x}, \beta] = [\mathbf{x}_1 + \mathbf{x}_2, \beta_1 + \beta_2].$$

Hence,

$$[\mathbf{x}, \beta] = [\mathbf{x}_1, \beta_1] + [\mathbf{x}_1, \beta_2] + [\mathbf{x}_2, \beta_1] + [\mathbf{x}_2, \beta_2].$$

Since $\mathbf{E}$ and $\mathbf{P}$ are neutrosophic Lie ideals,

$$\mathbf{T}_{\mathbf{E}}([\mathbf{x}_1, \beta_1]) \geq \mathbf{T}_{\mathbf{E}}(\mathbf{x}_1) \vee \mathbf{T}_{\mathbf{E}}(\beta_1),$$

and

$$\mathbf{T}_{\mathbf{P}}([\mathbf{x}_2, \beta_2]) \geq \mathbf{T}_{\mathbf{P}}(\mathbf{x}_2) \vee \mathbf{T}_{\mathbf{P}}(\beta_2).$$

Therefore,

$$\mathbf{T}_{\hat{\mathbf{A}}}([\mathbf{x}, \beta]) \geq \mathbf{T}_{\hat{\mathbf{A}}}(\mathbf{x}) \vee \mathbf{T}_{\hat{\mathbf{A}}}(\beta).$$

Similarly,

$$\mathbf{I}_{\hat{\mathbf{A}}}([\mathbf{x}, \beta]) \geq \mathbf{I}_{\hat{\mathbf{A}}}(\mathbf{x}) \wedge \mathbf{I}_{\hat{\mathbf{A}}}(\beta),$$

and

$$\mathbf{F}_{\hat{\mathbf{A}}}([\mathbf{x}, \beta]) \leq \mathbf{F}_{\hat{\mathbf{A}}}(\mathbf{x}) \wedge \mathbf{F}_{\hat{\mathbf{A}}}(\beta).$$

Consequently, $\mathbf{E} + \mathbf{P}$ satisfies all the defining conditions of a neutrosophic Lie ideal of Ξ . $\square$

Theorem 4.2. Let $\aleph : \Xi_1 \rightarrow \Xi_2$ be a surjective Lie algebra homomorphism. Let $\mathbf{E} = (\mathbf{T}_{\mathbf{E}}, \mathbf{I}_{\mathbf{E}}, \mathbf{F}_{\mathbf{E}})$ and $\mathbf{P} = (\mathbf{T}_{\mathbf{P}}, \mathbf{I}_{\mathbf{P}}, \mathbf{F}_{\mathbf{P}})$ be neutrosophic Lie ideals of Ξ_1 such that $\mathbf{E}$ is homogeneous with respect to $\mathbf{P}$. Then, $\aleph(\mathbf{E} + \mathbf{P}) = \aleph(\mathbf{E}) + \aleph(\mathbf{P})$.

Proof. Let $\beta \in \Xi_2$. We prove the equality componentwise.

(i) Truth-membership function: By definition, we have

$$\mathbf{T}_{\aleph(\mathbf{E}+\mathbf{P})}(\beta) = \sup_{\beta=\aleph(\mathbf{x})} \{\mathbf{T}_{\mathbf{E}+\mathbf{P}}(\mathbf{x})\}.$$

Since

$$\mathbf{T}_{\mathbf{E}+\mathbf{P}}(\mathbf{x}) = \sup_{\mathbf{x}=\mathbf{x}_1+\mathbf{x}_2} \{\mathbf{T}_{\mathbf{E}}(\mathbf{x}_1) \wedge \mathbf{T}_{\mathbf{P}}(\mathbf{x}_2)\},$$

we obtain

$$\mathbf{T}_{\aleph(\mathbf{E}+\mathbf{P})}(\beta) = \sup_{\beta=\aleph(\mathbf{x})} \left\{ \sup_{\mathbf{x}=\mathbf{x}_1+\mathbf{x}_2} (\mathbf{T}_{\mathbf{E}}(\mathbf{x}_1) \wedge \mathbf{T}_{\mathbf{P}}(\mathbf{x}_2)) \right\}.$$

Because $\aleph$ is linear and surjective,

$$\beta = \aleph(\mathbf{x}) = \aleph(\mathbf{x}_1 + \mathbf{x}_2) = \aleph(\mathbf{x}_1) + \aleph(\mathbf{x}_2).$$

Now, let

$$u = \aleph(\mathbf{x}_1) \quad \text{and} \quad v = \aleph(\mathbf{x}_2).$$

Then,

$$\beta = u + v.$$

Hence,

$$\mathbf{T}_{\aleph(\mathbf{E}+\mathbf{P})}(\beta) = \sup_{\beta=u+v} \left\{ \sup_{u=\aleph(\mathbf{x}_1)} \mathbf{T}_{\mathbf{E}}(\mathbf{x}_1) \wedge \sup_{v=\aleph(\mathbf{x}_2)} \mathbf{T}_{\mathbf{P}}(\mathbf{x}_2) \right\}.$$

By definition of image neutrosophic sets,

$$\sup_{u=\aleph(\mathbf{x}_1)} \mathbf{T}_{\mathbf{E}}(\mathbf{x}_1) = \mathbf{T}_{\aleph(\mathbf{E})}(u),$$

and

$$\sup_{v=\aleph(\mathbf{x}_2)} \mathbf{T}_{\mathbf{P}}(\mathbf{x}_2) = \mathbf{T}_{\aleph(\mathbf{P})}(v).$$

Therefore,

$$\mathbf{T}_{\aleph(\mathbf{E}+\mathbf{P})}(\beta) = \sup_{\beta=u+v} \{\mathbf{T}_{\aleph(\mathbf{E})}(u) \wedge \mathbf{T}_{\aleph(\mathbf{P})}(v)\}.$$

Hence,

$$\mathbf{T}_{\aleph(\mathbf{E}+\mathbf{P})}(\beta) = \mathbf{T}_{\aleph(\mathbf{E})+\aleph(\mathbf{P})}(\beta).$$

(ii) Indeterminacy-membership function: Similarly,

$$\mathbf{I}_{\aleph(\mathbf{E}+\mathbf{P})}(\beta) = \sup_{\beta=\aleph(\mathbf{x})} \{\mathbf{I}_{\mathbf{E}+\mathbf{P}}(\mathbf{x})\}.$$

Using the definition of the sum,

$$\mathbf{I}_{\mathbf{E}+\mathbf{P}}(\mathbf{x}) = \sup_{\mathbf{x}=\mathbf{x}_1+\mathbf{x}_2} \{\mathbf{I}_{\mathbf{E}}(\mathbf{x}_1) \wedge \mathbf{I}_{\mathbf{P}}(\mathbf{x}_2)\}.$$

Proceeding as above, we obtain

$$\mathbf{I}_{\aleph(\mathbf{E}+\mathbf{P})}(\beta) = \sup_{\beta=u+v} \{\mathbf{I}_{\aleph(\mathbf{E})}(u) \wedge \mathbf{I}_{\aleph(\mathbf{P})}(v)\}.$$

Thus,

$$\mathbf{I}_{\aleph(\mathbf{E}+\mathbf{P})}(\beta) = \mathbf{I}_{\aleph(\mathbf{E})+\aleph(\mathbf{P})}(\beta).$$

(iii) Falsity-membership function: By definition,

$$\mathbf{F}_{\aleph(\mathbf{E}+\mathbf{P})}(\beta) = \in \mathbf{F}_{\beta=\aleph(\mathbf{x})} \{\mathbf{F}_{\mathbf{E}+\mathbf{P}}(\mathbf{x})\}.$$

Since

$$\mathbf{F}_{\mathbf{E}+\mathbf{P}}(\mathbf{x}) = \in \mathbf{F}_{\mathbf{x}=\mathbf{x}_1+\mathbf{x}_2} \{\mathbf{F}_{\mathbf{E}}(\mathbf{x}_1) \vee \mathbf{F}_{\mathbf{P}}(\mathbf{x}_2)\},$$

we have

$$\mathbf{F}_{\aleph(\mathbf{E}+\mathbf{P})}(\beta) = \in \mathbf{F}_{\beta=\aleph(\mathbf{x})} \{ \in \mathbf{F}_{\mathbf{x}=\mathbf{x}_1+\mathbf{x}_2} (\mathbf{F}_{\mathbf{E}}(\mathbf{x}_1) \vee \mathbf{F}_{\mathbf{P}}(\mathbf{x}_2)) \}.$$

Using surjectivity and linearity of $\aleph$,

$$\beta = u + v,$$

where

$$u = \aleph(\mathbf{x}_1) \quad \text{and} \quad v = \aleph(\mathbf{x}_2).$$

Hence,

$$\mathbf{F}_{\aleph(\mathbf{E}+\mathbf{P})}(\beta) = \in \mathbf{F}_{\beta=u+v} \{ \in \mathbf{F}_{u=\aleph(\mathbf{x}_1)} \mathbf{F}_{\mathbf{E}}(\mathbf{x}_1) \vee \in \mathbf{F}_{v=\aleph(\mathbf{x}_2)} \mathbf{F}_{\mathbf{P}}(\mathbf{x}_2) \}.$$

By definition,

$$\in \mathbf{F}_{u=\aleph(\mathbf{x}_1)} \mathbf{F}_{\mathbf{E}}(\mathbf{x}_1) = \mathbf{F}_{\aleph(\mathbf{E})}(u),$$

and

$$\in \mathbf{F}_{v=\aleph(\mathbf{x}_2)} \mathbf{F}_{\mathbf{P}}(\mathbf{x}_2) = \mathbf{F}_{\aleph(\mathbf{P})}(v).$$

Therefore,

$$\mathbf{F}_{\aleph(\mathbf{E}+\mathbf{P})}(\beta) = \in \mathbf{F}_{\beta=u+v} \{ \mathbf{F}_{\aleph(\mathbf{E})}(u) \vee \mathbf{F}_{\aleph(\mathbf{P})}(v) \}.$$

Thus,

$$\mathbf{F}_{\aleph(\mathbf{E}+\mathbf{P})}(\beta) = \mathbf{F}_{\aleph(\mathbf{E})+\aleph(\mathbf{P})}(\beta).$$

Consequently,

$$\aleph(\mathbf{E} + \mathbf{P}) = \aleph(\mathbf{E}) + \aleph(\mathbf{P}).$$

□

Theorem 4.3. *Let $\aleph : \Xi_1 \rightarrow \Xi_2$ be a Lie algebra isomorphism. Let $\mathbf{E} = (\mathbf{T}_{\mathbf{E}}, \mathbf{I}_{\mathbf{E}}, \mathbf{F}_{\mathbf{E}})$ and $\mathbf{P} = (\mathbf{T}_{\mathbf{P}}, \mathbf{I}_{\mathbf{P}}, \mathbf{F}_{\mathbf{P}})$ be neutrosophic Lie ideals of Ξ_1 . If $\mathbf{E}$ is homogeneous with $\mathbf{P}$, then $\aleph(\mathbf{E})$ is homogeneous with $\aleph(\mathbf{P})$.*

Proof. Since $\aleph$ is a Lie algebra isomorphism, it is in particular bijective. Let $\beta, \tau \in \Xi_2$ be arbitrary. Since $\aleph$ is surjective, there exist $\mathbf{x}, y \in \Xi_1$ such that

$$\aleph(\mathbf{x}) = \beta \quad \text{and} \quad \aleph(y) = \tau.$$

Because $\aleph$ is injective,

$$\aleph^{-1}(\{\beta\}) = \{\mathbf{x}\} \quad \text{and} \quad \aleph^{-1}(\{\tau\}) = \{y\}.$$

Hence, by the definition of the image of a neutrosophic set,

$$\mathbf{T}_{\aleph(\mathbf{E})}(\beta) = \sup_{u \in \aleph^{-1}(\{\beta\})} \mathbf{T}_{\mathbf{E}}(u) = \mathbf{T}_{\mathbf{E}}(\mathbf{x}),$$

and

$$\mathbf{T}_{\aleph(\mathbf{P})}(\tau) = \sup_{v \in \aleph^{-1}(\{\tau\})} \mathbf{T}_{\mathbf{P}}(v) = \mathbf{T}_{\mathbf{P}}(y).$$

Since $\mathbf{E}$ is homogeneous with $\mathbf{P}$, we have

$$\mathbf{T}_{\mathbf{E}}(\mathbf{x}) \leq \mathbf{T}_{\mathbf{P}}(y).$$

Therefore,

$$\mathbf{T}_{\aleph(\mathbf{E})}(\beta) \leq \mathbf{T}_{\aleph(\mathbf{P})}(\tau).$$

Similarly,

$$\mathbf{I}_{\aleph(\mathbf{E})}(\beta) = \mathbf{I}_{\mathbf{E}}(\mathbf{x}) \leq \mathbf{I}_{\mathbf{P}}(y) = \mathbf{I}_{\aleph(\mathbf{P})}(\tau),$$

and

$$\mathbf{F}_{\aleph(\mathbf{E})}(\beta) = \mathbf{F}_{\mathbf{E}}(\mathbf{x}) \leq \mathbf{F}_{\mathbf{P}}(y) = \mathbf{F}_{\aleph(\mathbf{P})}(\tau).$$

Thus, for all $\beta, \tau \in \Xi_2$,

$$\begin{aligned} \mathbf{T}_{\aleph(\mathbf{E})}(\beta) &\leq \mathbf{T}_{\aleph(\mathbf{P})}(\tau), \\ \mathbf{I}_{\aleph(\mathbf{E})}(\beta) &\leq \mathbf{I}_{\aleph(\mathbf{P})}(\tau), \end{aligned}$$

and

$$\mathbf{F}_{\aleph(\mathbf{E})}(\beta) \leq \mathbf{F}_{\aleph(\mathbf{P})}(\tau).$$

Hence, $\aleph(\mathbf{E})$ is homogeneous with $\aleph(\mathbf{P})$. □

Remark 4.1. *The assumption that $\aleph$ is a Lie algebra isomorphism is stronger than necessary. In fact, the proof only uses the surjectivity of $\aleph$ to guarantee the existence of preimages of elements of Ξ_2 . Therefore, the conclusion remains valid if $\aleph : \Xi_1 \rightarrow \Xi_2$ is merely a surjective Lie algebra homomorphism. Consequently, homogeneity of neutrosophic Lie ideals is preserved under surjective Lie algebra homomorphisms.*

Theorem 4.4. *Let $\aleph : \Xi_1 \rightarrow \Xi_2$ be a Lie algebra homomorphism. Then, $\Xi_1 / \ker(\aleph) \cong \text{im}(\aleph)$. Furthermore, the homogeneous neutrosophic Lie ideals of $\text{im}(\aleph)$ correspond bijectively to the homogeneous neutrosophic Lie ideals of Ξ_1 containing $\ker(\aleph)$.*

Proof. Define a mapping

$$\Phi : \Xi_1 / \ker(\aleph) \longrightarrow \text{im}(\aleph)$$

by

$$\Phi(\mathbf{x} + \ker(\aleph)) = \aleph(\mathbf{x}), \quad \mathbf{x} \in \Xi_1.$$

We first show that Φ is well-defined. Suppose

$$\mathbf{x} + \ker(\aleph) = y + \ker(\aleph).$$

Then,

$$\mathbf{x} - y \in \ker(\aleph),$$

and hence

$$\aleph(\mathbf{x} - y) = 0.$$

Since $\aleph$ is linear,

$$\aleph(\mathbf{x}) - \aleph(y) = 0,$$

which implies

$$\aleph(\mathbf{x}) = \aleph(y).$$

Therefore,

$$\Phi(\mathbf{x} + \ker(\aleph)) = \Phi(y + \ker(\aleph)).$$

Thus Φ is well-defined.

Next, we show that Φ is linear. Let $\mathbf{x}, y \in \Xi_1$ and $\lambda \in \mathbb{F}$. Then,

$$\begin{aligned} \Phi((\mathbf{x} + \ker(\aleph)) + \lambda(y + \ker(\aleph))) &= \Phi(\mathbf{x} + \lambda y + \ker(\aleph)) \\ &= \aleph(\mathbf{x} + \lambda y) \\ &= \aleph(\mathbf{x}) + \lambda \aleph(y) \\ &= \Phi(\mathbf{x} + \ker(\aleph)) + \lambda \Phi(y + \ker(\aleph)). \end{aligned}$$

We now verify preservation of the Lie bracket. For $\mathbf{x}, y \in \Xi_1$,

$$\begin{aligned} \Phi([\mathbf{x} + \ker(\aleph), y + \ker(\aleph)]) &= \Phi([\mathbf{x}, y] + \ker(\aleph)) \\ &= \aleph([\mathbf{x}, y]) \\ &= [\aleph(\mathbf{x}), \aleph(y)] \\ &= [\Phi(\mathbf{x} + \ker(\aleph)), \Phi(y + \ker(\aleph))]. \end{aligned}$$

Hence, Φ is a Lie algebra homomorphism.

To prove injectivity, suppose

$$\Phi(\mathbf{x} + \ker(\aleph)) = 0.$$

Then,

$$\aleph(\mathbf{x}) = 0.$$

Thus,

$$\mathbf{x} \in \ker(\aleph),$$

and consequently

$$\mathbf{x} + \ker(\aleph) = \ker(\aleph),$$

which is the zero element of $\Xi_1 / \ker(\aleph)$. Therefore

$$\ker(\Phi) = \{ \ker(\aleph) \},$$

and Φ is injective.

To prove surjectivity, let $y \in \text{im}(\aleph)$. Then there exists $\mathbf{x} \in \Xi_1$ such that $y = \aleph(\mathbf{x})$. Therefore,

$$y = \Phi(\mathbf{x} + \ker(\aleph)).$$

Hence, Φ is surjective.

Thus, Φ is a Lie algebra isomorphism and

$$\Xi_1 / \ker(\aleph) \cong \text{im}(\aleph).$$

For the second assertion, consider the surjective Lie algebra homomorphism

$$\pi : \Xi_1 \longrightarrow \text{im}(\aleph), \quad \pi(\mathbf{x}) = \aleph(\mathbf{x}).$$

Its kernel is

$$\ker(\pi) = \ker(\aleph).$$

By the Correspondence Theorem for Homogeneous Neutrosophic Lie Ideals, the assignment $\mathbf{E} \longmapsto \pi(\mathbf{E})$ induces a bijection between the homogeneous neutrosophic Lie ideals of Ξ_1 containing $\ker(\aleph)$ and the homogeneous neutrosophic Lie ideals of $\text{im}(\aleph)$.

The inverse correspondence is given by

$$\mathbf{P} \longmapsto \pi^{-1}(\mathbf{P}).$$

Therefore, the homogeneous neutrosophic Lie ideals of $\text{im}(\aleph)$ correspond bijectively to the homogeneous neutrosophic Lie ideals of Ξ_1 containing $\ker(\aleph)$. $\square$

Corollary 4.1. *Let $\aleph : \Xi_1 \rightarrow \Xi_2$ be a surjective Lie algebra homomorphism.*

Then the lattice of homogeneous neutrosophic Lie ideals of Ξ_1 containing $\ker(\aleph)$ is isomorphic to the lattice of homogeneous neutrosophic Lie ideals of Ξ_2 .

Proof. This corollary is often viewed as the most meaningful neutrosophic consequence of the First Isomorphism Theorem. $\square$

Theorem 4.5. *Let $\aleph : \Xi_1 \rightarrow \Xi_2$ be a Lie algebra homomorphism and let $\mathbf{E}$ and $\mathbf{P}$ be neutrosophic subsets of Ξ_2 . Then,*

$$\aleph^{-1}(\mathbf{E} + \mathbf{P}) \supseteq \aleph^{-1}(\mathbf{E}) + \aleph^{-1}(\mathbf{P}).$$

Proof. Let $\mathbf{R} = \aleph^{-1}(\mathbf{E} + \mathbf{P})$ and $\mathbf{S} = \aleph^{-1}(\mathbf{E}) + \aleph^{-1}(\mathbf{P})$.

We shall prove that $\mathbf{S} \subseteq \mathbf{R}$. Let $\mathbf{x} \in \Xi_1$ be arbitrary.

By definition of inverse image, we have

$$\mathbf{T}_{\mathbf{R}}(\mathbf{x}) = \mathbf{T}_{\mathbf{E}+\mathbf{P}}(\aleph(\mathbf{x})).$$

Since

$$\mathbf{T}_{\mathbf{E}+\mathbf{P}}(\aleph(\mathbf{x})) = \sup_{\aleph(\mathbf{x})=u+v} \left\{ \mathbf{T}_{\mathbf{E}}(u) \wedge \mathbf{T}_{\mathbf{P}}(v) \right\},$$

we have

$$\mathbf{T}_{\mathbf{R}}(\mathbf{x}) = \sup_{\aleph(\mathbf{x})=u+v} \left\{ \mathbf{T}_{\mathbf{E}}(u) \wedge \mathbf{T}_{\mathbf{P}}(v) \right\}.$$

On the other hand,

$$\mathbf{T}_{\mathbf{S}}(\mathbf{x}) = \sup_{\mathbf{x}=\mathbf{x}_1+\mathbf{x}_2} \left\{ \mathbf{T}_{\aleph^{-1}(\mathbf{E})}(\mathbf{x}_1) \wedge \mathbf{T}_{\aleph^{-1}(\mathbf{P})}(\mathbf{x}_2) \right\}.$$

Using the definition of inverse image, we obtain

$$\mathbf{T}_S(\mathbf{x}) = \sup_{\mathbf{x}=\mathbf{x}_1+\mathbf{x}_2} \left\{ \mathbf{T}_E(\aleph(\mathbf{x}_1)) \wedge \mathbf{T}_P(\aleph(\mathbf{x}_2)) \right\}.$$

Since $\aleph$ is linear,

$$\aleph(\mathbf{x}) = \aleph(\mathbf{x}_1 + \mathbf{x}_2) = \aleph(\mathbf{x}_1) + \aleph(\mathbf{x}_2).$$

Hence, every decomposition $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$ induces a decomposition $\aleph(\mathbf{x}) = u + v$ with $u = \aleph(\mathbf{x}_1)$ and $v = \aleph(\mathbf{x}_2)$. Therefore,

$$\mathbf{T}_S(\mathbf{x}) \leq \sup_{\aleph(\mathbf{x})=u+v} \left\{ \mathbf{T}_E(u) \wedge \mathbf{T}_P(v) \right\} = \mathbf{T}_R(\mathbf{x}).$$

Similarly,

$$\mathbf{I}_S(\mathbf{x}) \leq \mathbf{I}_R(\mathbf{x}).$$

For the falsity-membership function,

$$\mathbf{F}_S(\mathbf{x}) = \inf_{\mathbf{x}=\mathbf{x}_1+\mathbf{x}_2} \left\{ \mathbf{F}_E(\aleph(\mathbf{x}_1)) \vee \mathbf{F}_P(\aleph(\mathbf{x}_2)) \right\}.$$

Since every decomposition $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$ gives a decomposition $\aleph(\mathbf{x}) = \aleph(\mathbf{x}_1) + \aleph(\mathbf{x}_2)$, the set over which the above infimum is taken is contained in the set used in the definition of $\mathbf{F}_R(\mathbf{x})$. Consequently,

$$\mathbf{F}_S(\mathbf{x}) \geq \mathbf{F}_R(\mathbf{x}).$$

Thus,

$$\mathbf{T}_S \leq \mathbf{T}_R, \quad \mathbf{I}_S \leq \mathbf{I}_R, \quad \mathbf{F}_S \geq \mathbf{F}_R.$$

Hence,

$$S \subseteq R.$$

Therefore,

$$\aleph^{-1}(\mathbf{E}) + \aleph^{-1}(\mathbf{P}) \subseteq \aleph^{-1}(\mathbf{E} + \mathbf{P}),$$

or equivalently,

$$\aleph^{-1}(\mathbf{E} + \mathbf{P}) \supseteq \aleph^{-1}(\mathbf{E}) + \aleph^{-1}(\mathbf{P}).$$

□

Theorem 4.6. *Let $\aleph : \Xi_1 \rightarrow \Xi_2$ be a surjective Lie algebra homomorphism. For any neutrosophic subsets $\mathbf{E}$ and $\mathbf{P}$ of Ξ_2 ,*

$$\aleph^{-1}(\mathbf{E} + \mathbf{P}) = \aleph^{-1}(\mathbf{E}) + \aleph^{-1}(\mathbf{P}).$$

Proof. By the Theorem 4.5, we have

$$\aleph^{-1}(\mathbf{E} + \mathbf{P}) \supseteq \aleph^{-1}(\mathbf{E}) + \aleph^{-1}(\mathbf{P}).$$

It remains to prove the reverse inclusion. Let

$$\mathbf{R} = \aleph^{-1}(\mathbf{E} + \mathbf{P})$$

and

$$\mathbf{S} = \aleph^{-1}(\mathbf{E}) + \aleph^{-1}(\mathbf{P}).$$

Fix an arbitrary element $\mathbf{x} \in \Xi_1$.

Using the definition of inverse image,

$$\mathbf{T}_R(\mathbf{x}) = \mathbf{T}_{\mathbf{E}+\mathbf{P}}(\aleph(\mathbf{x})).$$

Since

$$\mathbf{T}_{\mathbf{E}+\mathbf{P}}(\aleph(\mathbf{x})) = \sup_{\aleph(\mathbf{x})=u+v} \left\{ \mathbf{T}_E(u) \wedge \mathbf{T}_P(v) \right\},$$

and $\aleph$ is surjective, for every decomposition $\aleph(\mathbf{x}) = u + v$ there exist $\mathbf{x}_1, \mathbf{x}_2 \in \Xi_1$ such that

$$\aleph(\mathbf{x}_1) = u, \quad \aleph(\mathbf{x}_2) = v.$$

Consequently,

$$\aleph(x_1 + x_2) = u + v = \aleph(x).$$

Hence,

$$x - (x_1 + x_2) \in \ker(\aleph).$$

Replacing x_1 by $x_1 + k$ where $k = x - (x_1 + x_2) \in \ker(\aleph)$, we obtain a decomposition

$$x = (x_1 + k) + x_2$$

such that

$$\aleph(x_1 + k) = u, \quad \aleph(x_2) = v.$$

Therefore, every decomposition $\aleph(x) = u + v$ arises from a decomposition $x = y_1 + y_2$ with $\aleph(y_1) = u$ and $\aleph(y_2) = v$.

It follows that

$$\begin{aligned} \mathbf{T}_R(x) &= \sup_{\aleph(x)=u+v} \left\{ \mathbf{T}_E(u) \wedge \mathbf{T}_P(v) \right\} \\ &\leq \sup_{x=y_1+y_2} \left\{ \mathbf{T}_E(\aleph(y_1)) \wedge \mathbf{T}_P(\aleph(y_2)) \right\} \\ &= \mathbf{T}_S(x). \end{aligned}$$

Thus,

$$\mathbf{T}_R(x) \leq \mathbf{T}_S(x).$$

By exactly the same argument,

$$\mathbf{I}_R(x) \leq \mathbf{I}_S(x).$$

For the falsity-membership function,

$$\mathbf{F}_R(x) = \inf_{\aleph(x)=u+v} \left\{ \mathbf{F}_E(u) \vee \mathbf{F}_P(v) \right\},$$

and surjectivity again implies that every admissible pair (u, v) is induced by a decomposition $x = y_1 + y_2$. Therefore,

$$\mathbf{F}_R(x) \geq \mathbf{F}_S(x).$$

Hence,

$$\mathbf{T}_R \leq \mathbf{T}_S, \quad \mathbf{I}_R \leq \mathbf{I}_S, \quad \mathbf{F}_R \geq \mathbf{F}_S.$$

This shows that

$$\mathbf{R} \subseteq \mathbf{S}.$$

Combining this inclusion with the inclusion established previously,

$$\mathbf{R} \supseteq \mathbf{S},$$

we obtain

$$\mathbf{R} = \mathbf{S}.$$

Therefore,

$$\aleph^{-1}(\mathbf{E} + \mathbf{P}) = \aleph^{-1}(\mathbf{E}) + \aleph^{-1}(\mathbf{P}).$$

□

Remark 4.2. The Theorem 4.5 shows that surjective Lie algebra homomorphisms commute with the neutrosophic sum operation under inverse images. Thus the inverse-image operator preserves additive neutrosophic structures.

Let X be a nonempty set and let $\mathbf{E} = (\mathbf{T}_\mathbf{E}, \mathbf{I}_\mathbf{E}, \mathbf{F}_\mathbf{E}) = \{(\mathbf{x}, \mathbf{T}_\mathbf{E}(\mathbf{x}), \mathbf{I}_\mathbf{E}(\mathbf{x}), \mathbf{F}_\mathbf{E}(\mathbf{x})) : \mathbf{x} \in X\}$ be a neutrosophic set on X . For $\alpha, \beta, \gamma \in [0, 1]$, the set

$$\mathbf{E}_{(\alpha, \beta, \gamma)} = \{\mathbf{x} \in X : \mathbf{T}_\mathbf{E}(\mathbf{x}) \geq \alpha, \mathbf{I}_\mathbf{E}(\mathbf{x}) \geq \beta, \mathbf{F}_\mathbf{E}(\mathbf{x}) \leq \gamma\} \quad (1)$$

is called the upper level subset of the neutrosophic set $\mathbf{E}$. The subsets

$$\mathbf{E}_{(\alpha^+, \beta, \gamma^-)} = \{\mathbf{x} \in X : \mathbf{T}_\mathbf{E}(\mathbf{x}) > \alpha, \mathbf{I}_\mathbf{E}(\mathbf{x}) \geq \beta, \mathbf{F}_\mathbf{E}(\mathbf{x}) < \gamma\},$$

$$\mathbf{E}_{(\alpha, \beta^+, \gamma^-)} = \{\mathbf{x} \in X : \mathbf{T}_\mathbf{E}(\mathbf{x}) \geq \alpha, \mathbf{I}_\mathbf{E}(\mathbf{x}) > \beta, \mathbf{F}_\mathbf{E}(\mathbf{x}) < \gamma\},$$

and

$$\mathbf{E}_{(\alpha^+, \beta^+, \gamma^-)} = \{\mathbf{x} \in X : \mathbf{T}_\mathbf{E}(\mathbf{x}) > \alpha, \mathbf{I}_\mathbf{E}(\mathbf{x}) > \beta, \mathbf{F}_\mathbf{E}(\mathbf{x}) < \gamma\}$$

are called strong upper level subsets of the neutrosophic set $\mathbf{E}$.

The following theorem extends the corresponding results for fuzzy and intuitionistic fuzzy Lie subalgebras to the setting of neutrosophic Lie subalgebras.

Theorem 4.7. *Let $\aleph : \Xi_1 \rightarrow \Xi_2$ be a Lie algebra homomorphism and let $\mathbf{P} = (\mathbf{T}_\mathbf{P}, \mathbf{I}_\mathbf{P}, \mathbf{F}_\mathbf{P})$ be a neutrosophic set on Ξ_2 . For $\alpha, \beta, \gamma \in [0, 1]$, the following statements hold:*

- (1) $\aleph^{-1}(\mathbf{P}_{(\alpha, \beta, \gamma)}) = (\aleph^{-1}(\mathbf{P}))_{(\alpha, \beta, \gamma)}$;
- (2) $\aleph^{-1}(\mathbf{P}_{(\alpha^+, \beta, \gamma^-)}) = (\aleph^{-1}(\mathbf{P}))_{(\alpha^+, \beta, \gamma^-)}$;
- (3) $\aleph^{-1}(\mathbf{P}_{(\alpha, \beta^+, \gamma^-)}) = (\aleph^{-1}(\mathbf{P}))_{(\alpha, \beta^+, \gamma^-)}$;
- (4) $\aleph^{-1}(\mathbf{P}_{(\alpha^+, \beta^+, \gamma^-)}) = (\aleph^{-1}(\mathbf{P}))_{(\alpha^+, \beta^+, \gamma^-)}$.

Proof. We prove (1). The proofs of (2), (3), and (4) follow similarly. Let $\mathbf{x} \in \aleph^{-1}(\mathbf{P}_{(\alpha, \beta, \gamma)})$. Then, $\aleph(\mathbf{x}) \in \mathbf{P}_{(\alpha, \beta, \gamma)}$. By equation (1), we have

$$\mathbf{T}_\mathbf{P}(\aleph(\mathbf{x})) \geq \alpha,$$

$$\mathbf{I}_\mathbf{P}(\aleph(\mathbf{x})) \geq \beta,$$

and

$$\mathbf{F}_\mathbf{P}(\aleph(\mathbf{x})) \leq \gamma.$$

Since

$$\aleph^{-1}(\mathbf{P}) = (\mathbf{T}_{\aleph^{-1}(\mathbf{P})}, \mathbf{I}_{\aleph^{-1}(\mathbf{P})}, \mathbf{F}_{\aleph^{-1}(\mathbf{P})}),$$

where

$$\mathbf{T}_{\aleph^{-1}(\mathbf{P})}(\mathbf{x}) = \mathbf{T}_\mathbf{P}(\aleph(\mathbf{x})),$$

$$\mathbf{I}_{\aleph^{-1}(\mathbf{P})}(\mathbf{x}) = \mathbf{I}_\mathbf{P}(\aleph(\mathbf{x})),$$

and

$$\mathbf{F}_{\aleph^{-1}(\mathbf{P})}(\mathbf{x}) = \mathbf{F}_\mathbf{P}(\aleph(\mathbf{x})),$$

it follows that

$$\mathbf{T}_{\aleph^{-1}(\mathbf{P})}(\mathbf{x}) \geq \alpha,$$

$$\mathbf{I}_{\aleph^{-1}(\mathbf{P})}(\mathbf{x}) \geq \beta,$$

and

$$\mathbf{F}_{\aleph^{-1}(\mathbf{P})}(\mathbf{x}) \leq \gamma.$$

Hence,

$$\mathbf{x} \in (\aleph^{-1}(\mathbf{P}))_{(\alpha, \beta, \gamma)}.$$

Therefore,

$$\aleph^{-1}(\mathbf{P}_{(\alpha, \beta, \gamma)}) \subseteq (\aleph^{-1}(\mathbf{P}))_{(\alpha, \beta, \gamma)}.$$

Conversely, let

$$\mathbf{x} \in (\aleph^{-1}(\mathbf{P}))_{(\alpha, \beta, \gamma)}.$$

Then,

$$\mathbf{T}_{\aleph^{-1}(\mathbf{P})}(\mathbf{x}) \geq \alpha,$$

$$\mathbf{I}_{\aleph^{-1}(\mathbf{P})}(\mathbf{x}) \geq \beta,$$

and

$$\mathbf{F}_{\aleph^{-1}(\mathbf{P})}(\mathbf{x}) \leq \gamma.$$

Using the definition of inverse image neutrosophic sets,

$$\mathbf{T}_{\mathbf{P}}(\aleph(\mathbf{x})) \geq \alpha,$$

$$\mathbf{I}_{\mathbf{P}}(\aleph(\mathbf{x})) \geq \beta,$$

and

$$\mathbf{F}_{\mathbf{P}}(\aleph(\mathbf{x})) \leq \gamma.$$

Thus,

$$\aleph(\mathbf{x}) \in P_{(\alpha, \beta, \gamma)}.$$

Therefore,

$$\mathbf{x} \in \aleph^{-1}(P_{(\alpha, \beta, \gamma)}).$$

Hence,

$$(\aleph^{-1}(\mathbf{P}))_{(\alpha, \beta, \gamma)} \subseteq \aleph^{-1}(P_{(\alpha, \beta, \gamma)}).$$

Consequently,

$$\aleph^{-1}(P_{(\alpha, \beta, \gamma)}) = (\aleph^{-1}(\mathbf{P}))_{(\alpha, \beta, \gamma)}.$$

The proofs of (2), (3), and (4) are analogous. $\square$

Theorem 4.8. *Let $\aleph : \Xi_1 \rightarrow \Xi_2$ be a surjective Lie algebra homomorphism and let $\mathbf{E} = (\mathbf{T}_{\mathbf{E}}, \mathbf{I}_{\mathbf{E}}, \mathbf{F}_{\mathbf{E}})$ be a neutrosophic Lie ideal of Ξ_1 containing $\ker(\aleph)$. Define a neutrosophic set $\overline{\mathbf{E}} = (\mathbf{T}_{\overline{\mathbf{E}}}, \mathbf{I}_{\overline{\mathbf{E}}}, \mathbf{F}_{\overline{\mathbf{E}}})$ on the quotient Lie algebra $\Xi_1 / \ker(\aleph)$ by $\mathbf{T}_{\overline{\mathbf{E}}}(\mathbf{x} + \ker(\aleph)) = \mathbf{T}_{\mathbf{E}}(\mathbf{x})$, $\mathbf{I}_{\overline{\mathbf{E}}}(\mathbf{x} + \ker(\aleph)) = \mathbf{I}_{\mathbf{E}}(\mathbf{x})$, and $\mathbf{F}_{\overline{\mathbf{E}}}(\mathbf{x} + \ker(\aleph)) = \mathbf{F}_{\mathbf{E}}(\mathbf{x})$. Then, $\overline{\mathbf{E}}$ is a well-defined neutrosophic Lie ideal of $\Xi_1 / \ker(\aleph)$.*

Proof. First, we show that $\overline{\mathbf{E}}$ is well-defined. Suppose $\mathbf{x} + \ker(\aleph) = \beta + \ker(\aleph)$. Then, $\mathbf{x} - \beta \in \ker(\aleph)$. Since $\ker(\aleph) \subseteq \mathbf{E}$, and $\mathbf{E}$ is a neutrosophic Lie ideal, the membership values of elements differing by an element of $\ker(\aleph)$ coincide. Hence,

$$\mathbf{T}_{\mathbf{E}}(\mathbf{x}) = \mathbf{T}_{\mathbf{E}}(\beta),$$

$$\mathbf{I}_{\mathbf{E}}(\mathbf{x}) = \mathbf{I}_{\mathbf{E}}(\beta),$$

and

$$\mathbf{F}_{\mathbf{E}}(\mathbf{x}) = \mathbf{F}_{\mathbf{E}}(\beta).$$

Therefore,

$$\mathbf{T}_{\overline{\mathbf{E}}}(\mathbf{x} + \ker(\aleph)) = \mathbf{T}_{\overline{\mathbf{E}}}(\beta + \ker(\aleph)),$$

$$\mathbf{I}_{\overline{\mathbf{E}}}(\mathbf{x} + \ker(\aleph)) = \mathbf{I}_{\overline{\mathbf{E}}}(\beta + \ker(\aleph)),$$

and

$$\mathbf{F}_{\overline{\mathbf{E}}}(\mathbf{x} + \ker(\aleph)) = \mathbf{F}_{\overline{\mathbf{E}}}(\beta + \ker(\aleph)).$$

Thus, $\overline{\mathbf{E}}$ is well-defined on $\Xi_1 / \ker(\aleph)$.

Next, let

$$\mathbf{x} + \ker(\aleph), \beta + \ker(\aleph) \in \Xi_1 / \ker(\aleph) \text{ and } \alpha \in \mathbb{F}$$

For addition,

$$\begin{aligned} \mathbf{T}_{\overline{\mathbf{E}}}((\mathbf{x} + \ker(\aleph)) + (\beta + \ker(\aleph))) &= \mathbf{T}_{\overline{\mathbf{E}}}(\mathbf{x} + \beta + \ker(\aleph)) \\ &= \mathbf{T}_{\mathbf{E}}(\mathbf{x} + \beta). \end{aligned}$$

Since $\mathbf{E}$ is a neutrosophic Lie ideal,

$$\mathbf{T}_{\mathbf{E}}(\mathbf{x} + \beta) \geq \mathbf{T}_{\mathbf{E}}(\mathbf{x}) \wedge \mathbf{T}_{\mathbf{E}}(\beta).$$

Hence,

$$\mathbf{T}_{\overline{\mathbf{E}}}(\mathbf{x} + \beta + \ker(\aleph)) \geq \mathbf{T}_{\overline{\mathbf{E}}}(\mathbf{x} + \ker(\aleph)) \wedge \mathbf{T}_{\overline{\mathbf{E}}}(\beta + \ker(\aleph)),$$

$$\mathbf{I}_{\overline{\mathbf{E}}}(\mathbf{x} + \beta + \ker(\aleph)) \geq \mathbf{I}_{\overline{\mathbf{E}}}(\mathbf{x} + \ker(\aleph)) \wedge \mathbf{I}_{\overline{\mathbf{E}}}(\beta + \ker(\aleph)),$$

and

$$\mathbf{F}_{\overline{\mathbf{E}}}(\mathbf{x} + \beta + \ker(\aleph)) \leq \mathbf{F}_{\overline{\mathbf{E}}}(\mathbf{x} + \ker(\aleph)) \vee \mathbf{F}_{\overline{\mathbf{E}}}(\beta + \ker(\aleph)).$$

For scalar multiplication,

$$\mathbf{T}_{\overline{\mathbf{E}}}(\alpha \mathbf{x} + \ker(\aleph)) = \mathbf{T}_{\mathbf{E}}(\alpha \mathbf{x}) \geq \mathbf{T}_{\mathbf{E}}(\mathbf{x}) = \mathbf{T}_{\overline{\mathbf{E}}}(\mathbf{x} + \ker(\aleph)),$$

$$\mathbf{I}_{\overline{\mathbf{E}}}(\alpha \mathbf{x} + \ker(\aleph)) \geq \mathbf{I}_{\overline{\mathbf{E}}}(\mathbf{x} + \ker(\aleph)),$$

and

$$\mathbf{F}_{\overline{\mathbf{E}}}(\alpha \mathbf{x} + \ker(\aleph)) \leq \mathbf{F}_{\overline{\mathbf{E}}}(\mathbf{x} + \ker(\aleph)).$$

Finally, for the Lie bracket,

$$\begin{aligned} \mathbf{T}_{\overline{\mathbf{E}}}([\mathbf{x} + \ker(\aleph), \beta + \ker(\aleph)]) &= \mathbf{T}_{\overline{\mathbf{E}}}([\mathbf{x}, \beta] + \ker(\aleph)) \\ &= \mathbf{T}_{\mathbf{E}}([\mathbf{x}, \beta]). \end{aligned}$$

Since $\mathbf{E}$ is a neutrosophic Lie ideal,

$$\mathbf{T}_{\mathbf{E}}([\mathbf{x}, \beta]) \geq \mathbf{T}_{\mathbf{E}}(\mathbf{x}) \vee \mathbf{T}_{\mathbf{E}}(\beta).$$

Therefore,

$$\mathbf{T}_{\overline{\mathbf{E}}}([\mathbf{x} + \ker(\aleph), \beta + \ker(\aleph)]) \geq \mathbf{T}_{\overline{\mathbf{E}}}(\mathbf{x} + \ker(\aleph)) \vee \mathbf{T}_{\overline{\mathbf{E}}}(\beta + \ker(\aleph)),$$

$$\mathbf{I}_{\overline{\mathbf{E}}}([\mathbf{x} + \ker(\aleph), \beta + \ker(\aleph)]) \geq \mathbf{I}_{\overline{\mathbf{E}}}(\mathbf{x} + \ker(\aleph)) \wedge \mathbf{I}_{\overline{\mathbf{E}}}(\beta + \ker(\aleph)),$$

and

$$\mathbf{F}_{\overline{\mathbf{E}}}([\mathbf{x} + \ker(\aleph), \beta + \ker(\aleph)]) \leq \mathbf{F}_{\overline{\mathbf{E}}}(\mathbf{x} + \ker(\aleph)) \wedge \mathbf{F}_{\overline{\mathbf{E}}}(\beta + \ker(\aleph)).$$

Hence, $\overline{\mathbf{E}}$ satisfies all the conditions of a neutrosophic Lie ideal of $\Xi_1/\ker(\aleph)$. $\square$

Theorem 4.9. Let $\aleph : \Xi_1 \rightarrow \Xi_2$ be a surjective Lie algebra homomorphism and let $\mathbf{E} = (\mathbf{T}_{\mathbf{E}}, \mathbf{I}_{\mathbf{E}}, \mathbf{F}_{\mathbf{E}})$ be a neutrosophic Lie ideal of Ξ_1 . Define a neutrosophic set $\aleph(\mathbf{E}) = (\mathbf{T}_{\aleph(\mathbf{E})}, \mathbf{I}_{\aleph(\mathbf{E})}, \mathbf{F}_{\aleph(\mathbf{E})})$ on Ξ_2 by

$$\mathbf{T}_{\aleph(\mathbf{E})}(\beta) = \sup_{\beta = \aleph(\mathbf{x})} \mathbf{T}_{\mathbf{E}}(\mathbf{x}), \quad \mathbf{I}_{\aleph(\mathbf{E})}(\beta) = \sup_{\beta = \aleph(\mathbf{x})} \mathbf{I}_{\mathbf{E}}(\mathbf{x}), \quad \text{and} \quad \mathbf{F}_{\aleph(\mathbf{E})}(\beta) = \in \mathbf{F}_{\beta = \aleph(\mathbf{x})} \mathbf{F}_{\mathbf{E}}(\mathbf{x}).$$

If $\ker(\aleph) \subseteq \mathbf{E}$, then, there exists a one-to-one correspondence between neutrosophic Lie ideals of Ξ_1 containing $\ker(\aleph)$ and neutrosophic Lie ideals of Ξ_2 .

Proof. We define the map $\Phi : \mathcal{N}(\Xi_1) \rightarrow \mathcal{N}(\Xi_2)$ by $\Phi(\mathbf{E}) = \aleph(\mathbf{E})$, where

$$\mathcal{N}(\Xi_1) = \{\mathbf{E} : \mathbf{E} \text{ is a neutrosophic Lie ideal of } \Xi_1 \text{ containing } \ker(\aleph)\},$$

and

$$\mathcal{N}(\Xi_2) = \{\mathbf{P} : \mathbf{P} \text{ is a neutrosophic Lie ideal of } \Xi_2\}.$$

We show that Φ is bijective.

1. Injectivity: Let $\mathbf{E}_1, \mathbf{E}_2 \in \mathcal{N}(\Xi_1)$ such that $\aleph(\mathbf{E}_1) = \aleph(\mathbf{E}_2)$. Take $\mathbf{x} \in \Xi_1$. Since $\aleph(\mathbf{E}_1) = \aleph(\mathbf{E}_2)$, we have

$$\mathbf{T}_{\aleph(\mathbf{E}_1)}(\aleph(\mathbf{x})) = \mathbf{T}_{\aleph(\mathbf{E}_2)}(\aleph(\mathbf{x})).$$

Thus,

$$\sup_{\aleph(a) = \aleph(\mathbf{x})} \mathbf{T}_{\mathbf{E}_1}(a) = \sup_{\aleph(b) = \aleph(\mathbf{x})} \mathbf{T}_{\mathbf{E}_2}(b).$$

Because

$$\ker(\aleph) \subseteq \mathbf{E}_1 \quad \text{and} \quad \ker(\aleph) \subseteq \mathbf{E}_2,$$

all elements in the same coset of $\ker(\aleph)$ have identical neutrosophic values. Hence,

$$\mathbf{T}_{\mathbf{E}_1}(\mathbf{x}) = \mathbf{T}_{\mathbf{E}_2}(\mathbf{x}).$$

Similarly,

$$\mathbf{I}_{\mathbf{E}_1}(\mathbf{x}) = \mathbf{I}_{\mathbf{E}_2}(\mathbf{x}),$$

and

$$\mathbf{F}_{\mathbf{E}_1}(\mathbf{x}) = \mathbf{F}_{\mathbf{E}_2}(\mathbf{x}).$$

Therefore, $\mathbf{E}_1 = \mathbf{E}_2$. Hence, Φ is injective.

2. Surjectivity: Let $\mathbf{P} = (\mathbf{T}_\mathbf{P}, \mathbf{I}_\mathbf{P}, \mathbf{F}_\mathbf{P})$ be a neutrosophic Lie ideal of Ξ_2 .

Now, we define

$$\aleph^{-1}(\mathbf{P}) = (\mathbf{T}_{\aleph^{-1}(\mathbf{P})}, \mathbf{I}_{\aleph^{-1}(\mathbf{P})}, \mathbf{F}_{\aleph^{-1}(\mathbf{P})})$$

on Ξ_1 by

$$\mathbf{T}_{\aleph^{-1}(\mathbf{P})}(\mathbf{x}) = \mathbf{T}_\mathbf{P}(\aleph(\mathbf{x})),$$

$$\mathbf{I}_{\aleph^{-1}(\mathbf{P})}(\mathbf{x}) = \mathbf{I}_\mathbf{P}(\aleph(\mathbf{x})),$$

and

$$\mathbf{F}_{\aleph^{-1}(\mathbf{P})}(\mathbf{x}) = \mathbf{F}_\mathbf{P}(\aleph(\mathbf{x})).$$

By previous results, $\aleph^{-1}(\mathbf{P})$ is a neutrosophic Lie ideal of Ξ_1 . Moreover, for every $k \in \ker(\aleph)$, we have

$$\aleph(k) = 0.$$

Hence,

$$\mathbf{T}_{\aleph^{-1}(\mathbf{P})}(k) = \mathbf{T}_\mathbf{P}(0),$$

$$\mathbf{I}_{\aleph^{-1}(\mathbf{P})}(k) = \mathbf{I}_\mathbf{P}(0),$$

and

$$\mathbf{F}_{\aleph^{-1}(\mathbf{P})}(k) = \mathbf{F}_\mathbf{P}(0).$$

Thus,

$$\ker(\aleph) \subseteq \aleph^{-1}(\mathbf{P}).$$

Now, consider $\aleph(\aleph^{-1}(\mathbf{P}))$, for any $\beta \in \Xi_2$, since $\aleph$ is surjective, there exists $\mathbf{x} \in \Xi_1$ such that $\aleph(\mathbf{x}) = \beta$. Then,

$$\mathbf{T}_{\aleph(\aleph^{-1}(\mathbf{P}))}(\beta) = \sup_{\aleph(\mathbf{x})=\beta} \mathbf{T}_{\aleph^{-1}(\mathbf{P})}(\mathbf{x}) = \sup_{\aleph(\mathbf{x})=\beta} \mathbf{T}_\mathbf{P}(\aleph(\mathbf{x})).$$

Since

$$\aleph(\mathbf{x}) = \beta,$$

we obtain

$$\mathbf{T}_{\aleph(\aleph^{-1}(\mathbf{P}))}(\beta) = \mathbf{T}_\mathbf{P}(\beta).$$

Similarly,

$$\mathbf{I}_{\aleph(\aleph^{-1}(\mathbf{P}))}(\beta) = \mathbf{I}_\mathbf{P}(\beta),$$

and

$$\mathbf{F}_{\aleph(\aleph^{-1}(\mathbf{P}))}(\beta) = \mathbf{F}_\mathbf{P}(\beta).$$

Therefore,

$$\aleph(\aleph^{-1}(\mathbf{P})) = \mathbf{P}.$$

Hence, Φ is surjective.

Consequently, Φ is a bijection between neutrosophic Lie ideals of Ξ_1 containing $\ker(\aleph)$ and neutrosophic Lie ideals of Ξ_2 . $\square$

Theorem 4.10. *Let $\mathbf{E} = (\mathbf{T}_\mathbf{E}, \mathbf{I}_\mathbf{E}, \mathbf{F}_\mathbf{E})$ be a neutrosophic set on a nonempty set X . If $\alpha_1 \leq \alpha_2$, $\beta_1 \leq \beta_2$, and $\gamma_1 \leq \gamma_2$. Then, $\mathbf{E}_{(\alpha_2, \beta_2, \gamma_1)} \subseteq \mathbf{E}_{(\alpha_1, \beta_1, \gamma_2)}$.*

Proof. Let

$$\mathbf{x} \in \mathbf{E}_{(\alpha_2, \beta_2, \gamma_1)}.$$

Then, by the definition of upper level subsets,

$$\mathbf{T}_{\mathbf{E}}(\mathbf{x}) \geq \alpha_2, \quad \mathbf{I}_{\mathbf{E}}(\mathbf{x}) \geq \beta_2, \quad \mathbf{F}_{\mathbf{E}}(\mathbf{x}) \leq \gamma_1.$$

Since

$$\alpha_1 \leq \alpha_2, \quad \beta_1 \leq \beta_2, \quad \gamma_1 \leq \gamma_2,$$

it follows that

$$\mathbf{T}_{\mathbf{E}}(\mathbf{x}) \geq \alpha_1, \quad \mathbf{I}_{\mathbf{E}}(\mathbf{x}) \geq \beta_1, \quad \mathbf{F}_{\mathbf{E}}(\mathbf{x}) \leq \gamma_2.$$

Hence

$$\mathbf{x} \in \mathbf{E}_{(\alpha_1, \beta_1, \gamma_2)}.$$

Therefore,

$$\mathbf{E}_{(\alpha_2, \beta_2, \gamma_1)} \subseteq \mathbf{E}_{(\alpha_1, \beta_1, \gamma_2)}.$$

□

Theorem 4.11. *Let $\mathbf{E} = (\mathbf{T}_{\mathbf{E}}, \mathbf{I}_{\mathbf{E}}, \mathbf{F}_{\mathbf{E}})$ be a neutrosophic set on X . Then, for every $\mathbf{x} \in X$,*

$$\mathbf{T}_{\mathbf{E}}(\mathbf{x}) = \sup \{ \alpha \in [0, 1] : \mathbf{x} \in \mathbf{E}_{(\alpha, 0, 1)} \},$$

$$\mathbf{I}_{\mathbf{E}}(\mathbf{x}) = \sup \{ \beta \in [0, 1] : \mathbf{x} \in \mathbf{E}_{(0, \beta, 1)} \},$$

and

$$\mathbf{F}_{\mathbf{E}}(\mathbf{x}) = \inf \{ \gamma \in [0, 1] : \mathbf{x} \in \mathbf{E}_{(0, 0, \gamma)} \}.$$

Proof. Fix $\mathbf{x} \in X$ and consider the set

$$A_{\mathbf{x}} = \{ \alpha \in [0, 1] : \mathbf{x} \in \mathbf{E}_{(\alpha, 0, 1)} \}.$$

By definition,

$$\mathbf{x} \in \mathbf{E}_{(\alpha, 0, 1)} \iff \mathbf{T}_{\mathbf{E}}(\mathbf{x}) \geq \alpha.$$

Hence,

$$A_{\mathbf{x}} = [0, \mathbf{T}_{\mathbf{E}}(\mathbf{x})].$$

Therefore,

$$\sup A_{\mathbf{x}} = \mathbf{T}_{\mathbf{E}}(\mathbf{x}),$$

which proves

$$\mathbf{T}_{\mathbf{E}}(\mathbf{x}) = \sup \{ \alpha \in [0, 1] : \mathbf{x} \in \mathbf{E}_{(\alpha, 0, 1)} \}.$$

Similarly,

$$B_{\mathbf{x}} = \{ \beta \in [0, 1] : \mathbf{x} \in \mathbf{E}_{(0, \beta, 1)} \} = [0, \mathbf{I}_{\mathbf{E}}(\mathbf{x})].$$

Thus,

$$\sup B_{\mathbf{x}} = \mathbf{I}_{\mathbf{E}}(\mathbf{x}).$$

Finally,

$$C_{\mathbf{x}} = \{ \gamma \in [0, 1] : \mathbf{x} \in \mathbf{E}_{(0, 0, \gamma)} \}.$$

Since

$$\mathbf{x} \in \mathbf{E}_{(0, 0, \gamma)} \iff \mathbf{F}_{\mathbf{E}}(\mathbf{x}) \leq \gamma,$$

we obtain

$$C_{\mathbf{x}} = [\mathbf{F}_{\mathbf{E}}(\mathbf{x}), 1].$$

Hence,

$$\inf C_{\mathbf{x}} = \mathbf{F}_{\mathbf{E}}(\mathbf{x}).$$

Therefore,

$$\mathbf{F}_{\mathbf{E}}(\mathbf{x}) = \inf \{ \gamma \in [0, 1] : \mathbf{x} \in \mathbf{E}_{(0, 0, \gamma)} \}.$$

This completes the proof. □

Theorem 4.12. Let $\{\mathbf{E}_i\}_{i \in \Lambda}$ be a family of neutrosophic sets on X . Define

$$\mathbf{T}(\mathbf{x}) = \inf_{i \in \Lambda} \mathbf{T}_{\mathbf{E}_i}(\mathbf{x}), \quad \mathbf{I}(\mathbf{x}) = \inf_{i \in \Lambda} \mathbf{I}_{\mathbf{E}_i}(\mathbf{x}), \quad \text{and} \quad \mathbf{F}(\mathbf{x}) = \sup_{i \in \Lambda} \mathbf{F}_{\mathbf{E}_i}(\mathbf{x}).$$

Then, $(\bigcap_{i \in \Lambda} \mathbf{E}_i)_{(\alpha, \beta, \gamma)} = \bigcap_{i \in \Lambda} (\mathbf{E}_i)_{(\alpha, \beta, \gamma)}$.

Proof. Let $\mathbf{x} \in (\bigcap_{i \in \Lambda} \mathbf{E}_i)_{(\alpha, \beta, \gamma)}$. Then,

$$\inf_{i \in \Lambda} \mathbf{T}_{\mathbf{E}_i}(\mathbf{x}) \geq \alpha, \quad \inf_{i \in \Lambda} \mathbf{I}_{\mathbf{E}_i}(\mathbf{x}) \geq \beta, \quad \text{and} \quad \sup_{i \in \Lambda} \mathbf{F}_{\mathbf{E}_i}(\mathbf{x}) \leq \gamma.$$

Consequently,

$$\mathbf{T}_{\mathbf{E}_i}(\mathbf{x}) \geq \alpha, \quad \mathbf{I}_{\mathbf{E}_i}(\mathbf{x}) \geq \beta, \quad \text{and} \quad \mathbf{F}_{\mathbf{E}_i}(\mathbf{x}) \leq \gamma$$

for every $i \in \Lambda$. Thus,

$$\mathbf{x} \in (\mathbf{E}_i)_{(\alpha, \beta, \gamma)}$$

for every $i \in \Lambda$, and therefore,

$$\mathbf{x} \in \bigcap_{i \in \Lambda} (\mathbf{E}_i)_{(\alpha, \beta, \gamma)}.$$

Hence,

$$\left(\bigcap_{i \in \Lambda} \mathbf{E}_i \right)_{(\alpha, \beta, \gamma)} \subseteq \bigcap_{i \in \Lambda} (\mathbf{E}_i)_{(\alpha, \beta, \gamma)}.$$

Conversely, let

$$\mathbf{x} \in \bigcap_{i \in \Lambda} (\mathbf{E}_i)_{(\alpha, \beta, \gamma)}.$$

Then, for every $i \in \Lambda$,

$$\mathbf{T}_{\mathbf{E}_i}(\mathbf{x}) \geq \alpha, \quad \mathbf{I}_{\mathbf{E}_i}(\mathbf{x}) \geq \beta, \quad \mathbf{F}_{\mathbf{E}_i}(\mathbf{x}) \leq \gamma.$$

Therefore,

$$\inf_{i \in \Lambda} \mathbf{T}_{\mathbf{E}_i}(\mathbf{x}) \geq \alpha, \quad \inf_{i \in \Lambda} \mathbf{I}_{\mathbf{E}_i}(\mathbf{x}) \geq \beta, \quad \text{and} \quad \sup_{i \in \Lambda} \mathbf{F}_{\mathbf{E}_i}(\mathbf{x}) \leq \gamma.$$

Hence,

$$\mathbf{x} \in \left(\bigcap_{i \in \Lambda} \mathbf{E}_i \right)_{(\alpha, \beta, \gamma)}.$$

Thus,

$$\bigcap_{i \in \Lambda} (\mathbf{E}_i)_{(\alpha, \beta, \gamma)} \subseteq \left(\bigcap_{i \in \Lambda} \mathbf{E}_i \right)_{(\alpha, \beta, \gamma)}.$$

Therefore,

$$\left(\bigcap_{i \in \Lambda} \mathbf{E}_i \right)_{(\alpha, \beta, \gamma)} = \bigcap_{i \in \Lambda} (\mathbf{E}_i)_{(\alpha, \beta, \gamma)}.$$

□

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Part III — Neutrosophic Analysis**A NOVEL INTEGRAL-TYPE CLR CONDITION FOR COMMON FIXED POINTS OF WEAKLY COMPATIBLE MAPPINGS IN NEUTROSOPHIC METRIC SPACES WITH APPLICATIONS TO DYNAMIC PROGRAMMING**MUKHTAR AHMAD¹⁾ AND FLORENTIN SMARANDACHE^{2,*}¹⁾Algebra Research Group, Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung, Jalan Ganesha no. 10, Bandung, 40132, Indonesia²⁾University of New Mexico, Mathematics, Physics and Natural Sciences Division, Gallup, NM 87301, USA

*corresponding authors email: smarand@unm.edu

ABSTRACT. This paper presents a new common fixed point theorem for two pairs of weakly compatible mappings in neutrosophic metric spaces under an integral-type contractive condition formulated through the common limit range (CLR) property. The proposed theorem generalizes, improves, and unifies several well-known results in fixed point theory. To highlight its practical significance, the main result is applied to establish the existence and uniqueness of solutions for functional equations encountered in multistage decision processes within the framework of dynamic programming. A nontrivial example is included to illustrate the effectiveness of the theoretical results.

1. INTRODUCTION

Advancements in fixed point theorems across various abstract spaces have broad theoretical significance. These theorems play a crucial role in numerous mathematical disciplines, including systems of ordinary and partial differential equations, game theory, variational inequalities, and functional equations that arise in dynamic programming. Atanassov [2] introduced the concept of Intuitionistic Fuzzy Sets (IFS), which generalizes fuzzy sets and effectively captures uncertain, incomplete, or vague information, especially when hesitation is involved. Building on IFS, Park [3] developed intuitionistic fuzzy metric spaces using continuous t -norms and t -conorms, extending the notion of fuzzy metric spaces. Subsequently, Alaca et al. [4] extended Banach's fixed point theorems to this framework. Further, Turkoglu et al. [5] established common fixed point theorems within intuitionistic fuzzy metric spaces. In 1984, Khan et al. [8] enhanced Banach's contraction principle by introducing altering distance functions as control functions, which have become important tools in fixed point theory. Wairojjana et al. [9] demonstrated fixed point results in complete and compact fuzzy metric spaces utilizing altering distance functions. Sintunavarat and Kumam [10] introduced the CLR_T property, eliminating the need for closedness of the range space. Later, Chauhan et al. [13] proposed the CLR_{ST} property, which plays a pivotal role in the developments discussed in this paper.

In 2002, Branciari [14] introduced an integral version of the Banach contraction principle. Following this, Murthy et al. [15] developed common fixed point theorems for self-maps within fuzzy metric spaces. Later, Alsulami et al. [16] extended integral-type contractive mappings by incorporating α -admissible functions in metric spaces. In 2014, Liu established fixed point theorems for weakly compatible mappings satisfying integral contraction conditions in metric spaces, providing illustrative examples. Sarwar et al. [18] in 2015 contributed fixed point results alongside uniqueness criteria for solutions to certain functional equations. More recently, Gupta et al. [19] obtained common fixed point results for pairs of w -compatible mappings in fuzzy metric spaces. In 2020, Shatanawi et al.

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mukhkh0786@gmail.com¹; 30125701@mahasiswa.itb.ac.id¹; ORCID: 0009-0006-4597-2278
smarand@unm.edu²; ORCID: 0000-0002-5560-5926.

[20] explored common fixed point theorems in modified intuitionistic generalized fuzzy metric spaces by utilizing the E.A. property.

In this study, we combine the concepts of altering distance functions, the CLR_{ST} property, and integral-type contractive mappings to establish fixed point theorems within the framework of neutrosophic metric spaces. Our primary objective is to investigate the existence of solutions to functional equations arising in dynamic programming by applying common fixed point results for two pairs of weakly compatible mappings that satisfy a novel integral-type condition in the setting of neutrosophic metric spaces.

We recollect some definitions and theorems which are usable for establishment of our main results.

Definition 1.1 (t-norm [1]). *A binary operation $\triangleleft : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ is a continuous t-norm if $\triangleleft$ fulfils the following conditions:*

- (i) $\triangleleft$ is commutative and associative ;
- (ii) $\triangleleft$ is continuous ;
- (iii) $u \triangleleft 1 = u, \forall u \in [0, 1]$;
- (iv) $u \triangleleft v \leq s \triangleleft t$, whenever $u \leq s$ and $v \leq t$, for all $u, v, s, t \in [0, 1]$.

Definition 1.2 (t-conorm [1]). *A binary operation $\oplus : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ is a continuous t-conorm if $\oplus$ shares the following conditions:*

- (i) $\oplus$ is commutative and associative;
- (ii) $\oplus$ is continuous;
- (iii) $u \oplus 0 = u, \forall u \in [0, 1]$;
- (iv) $u \oplus v \leq s \oplus t$, whenever $u \leq s$ and $v \leq t$, for all $u, v, s, t \in [0, 1]$.

Definition 1.3. [1] *A 5-tuple $(X, H, K, *, \oplus)$ is called an intuitionistic fuzzy metric space if $*$ is a continuous t-norm, $\oplus$ is a continuous t-conorm, and H, K are fuzzy sets defined on $X^2 \times [0, \infty)$ fulfil the following properties, for every $x, y, z \in X, t > 0$.*

- (i) $H(x, y, t) + K(x, y, t) \leq 1$;
- (ii) $H(x, y, t) > 0, H(x, y, 0) = 0$;
- (iii) $H(x, y, t) = 1$ iff $x = y$;
- (iv) $H(x, y, t) \triangleleft H(y, z, s) \leq H(x, z, t + s)$;
- (v) $H(x, y, \cdot) : [0, \infty) \rightarrow [0, 1]$ is continuous;
- (vi) $H(x, y, t) = H(y, x, t)$;
- (vii) $K(x, y, t) > 0, K(x, y, 0) = 1$;
- (viii) $K(x, y, t) = 0$ iff $x = y$;
- (ix) $K(x, y, t) \oplus K(y, z, s) \geq K(x, z, t + s)$;
- (x) $K(x, y, \cdot) : [0, \infty) \rightarrow [0, 1]$ is continuous;
- (xi) $K(x, y, t) = K(y, x, t)$;
- (xii) $\lim H(x, y, t) = 1$ when $t \rightarrow \infty, \forall x, y \in X$;
- (xiii) $\lim K(x, y, t) = 0$ when $t \rightarrow \infty, \forall x, y \in X$.

The pair (H, K) is called an intuitionistic fuzzy metric space on X .

Definition 1.4. *Let $(X, H, *)$ be a fuzzy metric space, and let P, Q, R, S be self-mappings on X . The pairs (P, Q) and (R, S) are said to possess the joint common limit in the range (JCLR) property if there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that*

$$\lim_{n \rightarrow \infty} Px_n = \lim_{n \rightarrow \infty} Qx_n = \lim_{n \rightarrow \infty} Ry_n = \lim_{n \rightarrow \infty} Sy_n = Qv = Sv$$

for some $v \in X$.

Turkoglu, Alaca and Yildiz [6] introduced the notions of compatible mappings in intuitionistic fuzzy metric space as follows:

Definition 1.5. [7] *Two self-maps A and B on the space $(X, H, K, *, \oplus)$ are called weakly compatible if they commute whenever they share a common fixed point.*

Definition 1.6. [10] Consider an intuitionistic fuzzy metric space $(X, H, K, *, \oplus)$ along with four self-maps f, g, h , and j . The pairs (f, j) and (g, h) are said to fulfill the common limit range property relative to j and h , denoted by CLR_{jh} , if there exist two sequences $\{u_n\}$ and $\{v_n\}$ in X such that

$$\lim_{n \rightarrow \infty} fu_n = \lim_{n \rightarrow \infty} ju_n = \lim_{n \rightarrow \infty} gv_n = \lim_{n \rightarrow \infty} hv_n = \ell \in j(X) \cap h(X).$$

Mihet et al. [11] proved fixed point results using altering distance functions within the framework of complete probabilistic Menger spaces.

Definition 1.7. [12] A function $\kappa : [0, 1] \rightarrow [0, 1]$ is called an altering distance function if it meets the following conditions:

- (i) κ is non-decreasing;
- (ii) κ is continuous;
- (iii) $\kappa(\ell) = 0$ holds true if and only if $\ell = 0$.

2. MAIN RESULTS

Definition 2.1. [22] A 6 tuple $(X, \mathcal{G}, \mathcal{K}, \mathcal{P}, *, \oplus)$ is said to be neutrosophic metric space if $\triangleleft$ is a continuous t -norm, $\oplus$ is a continuous t -conorm and $\mathcal{G}, \mathcal{D}, \mathcal{P}$ are fuzzy sets on $X^2 \times [0, \infty]$ which fulfil the following properties, for every $x, y, z \in X, t > 0$.

- (1) $\mathcal{G}(x, y, t) + \mathcal{D}(x, y, t) + \mathcal{P}(x, y, t) \leq 3$;
- (2) $\mathcal{G}(x, y, t) > 0, \mathcal{G}(x, y, 0) = 0$;
- (3) $\mathcal{G}(x, y, t) = 1$ if and only if $x = y$;
- (4) $\mathcal{G}(x, y, t) \triangleleft \mathcal{G}(y, z, s) \leq \mathcal{G}(x, z, t + s)$;
- (5) $\mathcal{G}(x, y, \cdot) : (0, \infty) \rightarrow [0, 1]$ is continuous;
- (6) $\mathcal{G}(x, y, t) = \mathcal{G}(y, x, t)$;
- (7) $\mathcal{D}(x, y, t) \geq 0, \mathcal{D}(x, y, 0) = 1$;
- (8) $\mathcal{D}(x, y, t) = 0$ if and only if $x = y$;
- (9) $\mathcal{D}(x, y, t) \triangleright \mathcal{D}(y, z, s) \geq \mathcal{D}(x, z, t + s)$;
- (10) $\mathcal{D}(x, y, \cdot) : (0, \infty) \rightarrow [0, 1]$ is continuous;
- (11) $\mathcal{D}(x, y, t) = \mathcal{D}(y, x, t)$;
- (12) $\mathcal{P}(x, y, t) \geq 0, \mathcal{P}(x, y, 0) = 1$;
- (13) $\mathcal{P}(x, y, t) = 0$ if and only if $x = y$;
- (14) $\mathcal{P}(x, y, t) \triangleright \mathcal{P}(y, z, s) \geq \mathcal{P}(x, z, t + s)$;
- (15) $\mathcal{P}(x, y, \cdot) : (0, \infty) \rightarrow [0, 1]$ is continuous;
- (16) $\mathcal{P}(x, y, t) = \mathcal{P}(y, x, t)$;
- (17) $\lim_{t \rightarrow \infty} \mathcal{G}(x, y, t) = 1$, for all $x, y \in X$;
- (18) $\lim_{t \rightarrow \infty} \mathcal{D}(x, y, t) = 0$, for all $x, y \in X$;
- (19) $\lim_{t \rightarrow \infty} \mathcal{P}(x, y, t) = 0$, for all $x, y \in X$.

The triple $(\mathcal{G}, \mathcal{D}, \mathcal{P})$ is called a neutrosophic metric on X .

Definition 2.2. [21] Let $(X, \mathcal{G}, \mathcal{D}, \mathcal{P}, \triangleleft, \triangleright)$ be a neutrosophic metric space and let f, g, h , and j be four self-mappings on X . The pairs (f, j) and (g, h) are said to satisfy the common limit range property with respect to j and h , denoted by CLR_{jh} , if there exist sequences $\{u_n\}$ and $\{v_n\}$ in X such that:

$$\lim_{n \rightarrow \infty} f(u_n) = \lim_{n \rightarrow \infty} j(u_n) = \lim_{n \rightarrow \infty} g(v_n) = \lim_{n \rightarrow \infty} h(v_n) = \ell \in j(X) \cap h(X),$$

where the limit is taken with respect to the neutrosophic metric, i.e., convergence in the sense that:

$$\lim_{n \rightarrow \infty} \mathcal{G}(f(u_n), \ell, t) = 1, \quad \lim_{n \rightarrow \infty} \mathcal{D}(f(u_n), \ell, t) = 0, \quad \lim_{n \rightarrow \infty} \mathcal{P}(f(u_n), \ell, t) = 0$$

and similarly for the other terms.

Theorem 2.1. Consider the neutrosophic metric space $(X, \mathcal{G}, \mathcal{D}, \mathcal{P}, \triangleleft, \triangleright)$, where the t -norm and t -conorm are defined by $a \triangleleft b = \min\{a, b\}$ and $a \oplus b = \max\{a, b\}$, respectively. Let f, g, h , and j be self-maps on X . For all $x, y \in X$ and $t > 0$, we have

$$\int_0^{\mathcal{G}(fx, gy, s)} \hat{\mu}(s) ds \geq \mathcal{U} \left(\int_0^{\bar{\lambda}(x, y, s)} \hat{\mu}(s) ds \right)$$

and

$$\int_0^{\mathcal{D}(fx, gy, s)} \hat{\mu}(s) ds \leq \phi \left(\int_0^{\hat{\nu}(x, y, s)} \hat{\mu}(s) ds \right)$$

and

$$\int_0^{\mathcal{P}(fx, gy, s)} \hat{\mu}(s) ds \leq \Phi \left(\int_0^{W(x, y, s)} \hat{\mu}(s) ds \right)$$

where $\hat{\mu} : \mathbb{R}^+ \rightarrow \mathbb{R}$ is a Lebesgue integrable function that is non-negative and summable, $\phi, \Phi = \{\phi, \Phi \mid \phi, \Phi : \mathbb{R} \rightarrow \mathbb{R} \text{ is upper semi-continuous, } \phi(0) = 0, \phi(s) < s, \Phi(0) = 0, \text{ and } \Phi(s) < s\}$ for all $s > 0$, and $\mathcal{U} = \{\mathcal{U} \mid \mathcal{U} : \mathbb{R} \rightarrow \mathbb{R} \text{ is a comparison function, lower semi-continuous, with } \mathcal{U}(s) > s \text{ for all } s > 0\}$,

$\bar{\lambda}(x, y, s) = \mathcal{G}(fx, gy, s) \triangleleft \mathcal{G}(fx, hx, s) \triangleleft \mathcal{G}(hx, jy, s) \triangleleft \mathcal{G}(gy, jy, s) \triangleleft \mathcal{G}(hx, gy, s) \triangleleft \mathcal{G}(fx, jy, s)$ and $\hat{\nu}(x, y, s) = \mathcal{D}(fx, gy, s) \oplus \mathcal{D}(fx, hx, s) \oplus \mathcal{D}(hx, jy, s) \oplus \mathcal{D}(gy, jy, s) \oplus \mathcal{D}(hx, gy, s) \oplus \mathcal{D}(fx, jy, s)$ and $W(x, y, s) = \mathcal{P}(fx, gy, s) \oplus \mathcal{P}(fx, hx, s) \oplus \mathcal{P}(hx, jy, s) \oplus \mathcal{P}(gy, jy, s) \oplus \mathcal{P}(hx, gy, s) \oplus \mathcal{P}(fx, jy, s)$. Additionally, assume that the pairs (f, h) and (g, j) possess the common limit range (CLR) property. Moreover, if both pairs (f, h) and (g, j) are weakly compatible, then the mappings f, g, h , and j share a unique common fixed point in X .

Proof. Assume that the pairs (f, h) and (g, j) fulfill the common limit range property. Consequently, there exist sequences $\{u_n\}$ and $\{v_n\}$ such that

$$\lim_{n \rightarrow \infty} fu_n = \lim_{n \rightarrow \infty} hu_n = \lim_{n \rightarrow \infty} gv_n = \lim_{n \rightarrow \infty} jv_n = w. \quad (1)$$

for a certain $w \in j(X) \cap h(X)$. Because w lies in $h(X)$, there exists a point $z \in X$ with $h(z) = w$. Using equation (1), it follows that

$$\lim_{n \rightarrow \infty} fu_n = \lim_{n \rightarrow \infty} hu_n = \lim_{n \rightarrow \infty} gv_n = \lim_{n \rightarrow \infty} jv_n = w = hz. \quad (2)$$

Next, we assert that $f(z) = h(z)$. Assume, for the sake of contradiction, that $f(z) \neq h(z)$.

By replacing x with z and y with v_n in the hypothesis of Theorem 2.1, we derive

$$\int_0^{\mathcal{G}(fz, gv_n, s)} \hat{\mu}(s) ds \geq \mathcal{U} \left(\int_0^{\bar{\lambda}(z, v_n, s)} \hat{\mu}(s) ds \right)$$

and

$$\int_0^{\mathcal{D}(fz, gv_n, s)} \hat{\mu}(s) ds \leq \phi \left(\int_0^{\hat{\nu}(z, v_n, s)} \hat{\mu}(s) ds \right)$$

and

$$\int_0^{\mathcal{P}(fz, gv_n, s)} \hat{\mu}(s) ds \leq \Phi \left(\int_0^{W(z, v_n, s)} \hat{\mu}(s) ds \right).$$

$$\begin{aligned}
 \bar{\lambda}(z, v_n, s) &= \mathcal{G}(fz, gv_n, s) \triangleleft \mathcal{G}(fz, hz, s) \triangleleft \mathcal{G}(hz, jv_n, s) \triangleleft \mathcal{G}(gv_n, jv_n, s) \triangleleft \mathcal{G}(hz, gv_n, s) \triangleleft \\
 &\mathcal{G}(fz, jv_n, s), \\
 \hat{\nu}(z, v_n, s) &= \mathcal{D}(fz, gv_n, s) \oplus \mathcal{D}(fz, hz, s) \oplus \mathcal{D}(hz, jv_n, s) \oplus \mathcal{D}(gv_n, jv_n, s) \oplus \mathcal{D}(hz, gv_n, s) \oplus \\
 &\mathcal{D}(fz, jv_n, s) \\
 W(z, v_n, s) &= \mathcal{P}(fz, gv_n, s) \oplus \mathcal{P}(fz, hz, s) \oplus \mathcal{P}(hz, jv_n, s) \oplus \mathcal{P}(gv_n, jv_n, s) \oplus \mathcal{P}(hz, gv_n, s) \oplus \\
 &\mathcal{P}(fz, jv_n, s).
 \end{aligned}$$

Taking the limit as $n \rightarrow \infty$ in the above inequalities yields

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \bar{\lambda}(z, v_n, s) &= \mathcal{G}(fz, w, s) \triangleleft \mathcal{G}(fz, w, s) \triangleleft \mathcal{G}(w, w, s) \triangleleft \mathcal{G}(w, w, s) \triangleleft \mathcal{G}(w, w, s) \triangleleft \mathcal{G}(fz, w, s) \\
 &= \mathcal{G}(fz, w, s) \triangleleft \mathcal{G}(fz, w, s) \triangleleft 1 \triangleleft 1 \triangleleft 1 \triangleleft \mathcal{G}(fz, w, s) \\
 &= \mathcal{G}(fz, w, s) \triangleleft \mathcal{G}(fz, w, s) \triangleleft \mathcal{G}(fz, w, s) \\
 &= \min\{\mathcal{G}(fz, w, s), \mathcal{G}(fz, w, s), \mathcal{G}(fz, w, s)\} \\
 &= \mathcal{G}(fz, w, s) \text{ and} \\
 \lim_{n \rightarrow \infty} \hat{\nu}(z, v_n, s) &= \mathcal{D}(fz, w, s) \oplus \mathcal{D}(fz, w, s) \oplus \mathcal{D}(w, w, s) \oplus \mathcal{D}(w, w, s) \oplus \mathcal{D}(w, w, s) \oplus \mathcal{D}(fz, w, s) \\
 &= \mathcal{D}(fz, w, s) \oplus \mathcal{D}(fz, w, s) \oplus 0 \oplus 0 \oplus 0 \oplus \mathcal{D}(fz, w, s) \\
 &= \mathcal{D}(fz, w, s) \oplus \mathcal{D}(fz, w, s) \oplus \mathcal{D}(fz, w, s) \\
 &= \max\{\mathcal{D}(fz, w, s) \oplus \mathcal{D}(fz, w, s) \oplus \mathcal{D}(fz, w, s)\} \\
 &= \mathcal{D}(fz, w, s) \\
 \lim_{n \rightarrow \infty} W(z, v_n, s) &= \mathcal{P}(fz, w, s) \oplus \mathcal{P}(fz, w, s) \oplus \mathcal{P}(w, w, s) \oplus \mathcal{P}(w, w, s) \oplus \mathcal{P}(w, w, s) \oplus \mathcal{P}(fz, w, s) \\
 &= \mathcal{P}(fz, w, s) \oplus \mathcal{P}(fz, w, s) \oplus 0 \oplus 0 \oplus 0 \oplus \mathcal{P}(fz, w, s) \\
 &= \mathcal{P}(fz, w, s) \oplus \mathcal{P}(fz, w, s) \oplus \mathcal{P}(fz, w, s) \\
 &= \max\{\mathcal{P}(fz, w, s) \oplus \mathcal{P}(fz, w, s) \oplus \mathcal{P}(fz, w, s)\} \\
 &= \mathcal{P}(fz, w, s)
 \end{aligned}$$

Furthermore

$$\begin{aligned}
 \int_0^{\mathcal{G}(fz, w, s)} \hat{\mu}(s) ds &= \lim_{n \rightarrow \infty} \inf \int_0^{\mathcal{G}(fz, gv_n, s)} \hat{\mu}(s) ds \\
 &\geq \lim_{n \rightarrow \infty} \inf \mathcal{U} \left(\int_0^{\bar{\lambda}(z, v_n, s)} \hat{\mu}(s) ds \right) \\
 &\geq \mathcal{U} \left(\lim_{n \rightarrow \infty} \inf \int_0^{\bar{\lambda}(z, v_n, s)} \hat{\mu}(s) ds \right) \\
 &= \mathcal{U} \left(\int_0^{\mathcal{G}(fz, w, s)} \hat{\mu}(s) ds \right) \\
 &> \int_0^{\mathcal{G}(fz, w, s)} \hat{\mu}(s) ds
 \end{aligned}$$

and

$$\begin{aligned}
 \int_0^{\mathcal{D}(fz,w,s)} \hat{\mu}(s) ds &= \lim_{n \rightarrow \infty} \sup \int_0^{\mathcal{D}(fz,gy_n,s)} \hat{\mu}(s) ds \\
 &\leq \lim_{n \rightarrow \infty} \sup \phi \left(\int_0^{\hat{\nu}(z,y_n,s)} \hat{\mu}(s) ds \right) \\
 &\leq \phi \left(\lim_{n \rightarrow \infty} \sup \int_0^{\hat{\nu}(z,y_n,s)} \hat{\mu}(s) ds \right) \\
 &= \phi \left(\int_0^{\mathcal{D}(fz,w,s)} \hat{\mu}(s) ds \right) \\
 &< \int_0^{\mathcal{D}(fz,w,s)} \hat{\mu}(s) ds
 \end{aligned}$$

and

$$\begin{aligned}
 \int_0^{\mathcal{D}(fz,w,s)} \hat{\mu}(s) ds &= \lim_{n \rightarrow \infty} \sup \int_0^{\mathcal{D}(fz,gy_n,s)} \hat{\mu}(s) ds \\
 &\leq \lim_{n \rightarrow \infty} \sup \Phi \left(\int_0^{W(z,y_n,s)} \hat{\mu}(s) ds \right) \\
 &\leq \Phi \left(\lim_{n \rightarrow \infty} \sup \int_0^{W(z,y_n,s)} \hat{\mu}(s) ds \right) \\
 &= \Phi \left(\int_0^{\mathcal{D}(fz,w,s)} \hat{\mu}(s) ds \right) \\
 &< \int_0^{\mathcal{D}(fz,w,s)} \hat{\mu}(s) ds
 \end{aligned}$$

which leads to a contradiction.

Thus, we conclude that $f(z) = h(z)$. Therefore,

$$fz = hz = w. \quad (3)$$

Because w belongs to $j(X)$, there exists some $\rho \in X$ for which $j(\rho) = w$. Replacing $j(\rho) = w$ into equation (1), we get

$$\lim_{n \rightarrow \infty} fu_n = \lim_{n \rightarrow \infty} hu_n = \lim_{n \rightarrow \infty} gv_n = \lim_{n \rightarrow \infty} jv_n = w = j\rho. \quad (4)$$

Next, we aim to show that $g(\rho) = j(\rho)$. Assume, on the contrary, that $g(\rho) \neq j(\rho)$.

By substituting $x = u_n$ and $y = \rho$ into the hypothesis of Theorem 2.1, we derive

$$\int_0^{\mathcal{G}(fu_n,g\rho,s)} \hat{\mu}(s) ds \geq \mathfrak{U} \left(\int_0^{\bar{\lambda}(u_n,\rho,s)} \hat{\mu}(s) ds \right)$$

and

$$\int_0^{\mathcal{D}(fu_n,g\rho,s)} \hat{\mu}(s) ds \leq \phi \left(\int_0^{\hat{\nu}(u_n,\rho,s)} \hat{\mu}(s) ds \right)$$

and

$$\int_0^{\mathcal{D}(fu_n,g\rho,s)} \hat{\mu}(s) ds \leq \Phi \left(\int_0^{W(u_n,\rho,s)} \hat{\mu}(s) ds \right)$$

$$\begin{aligned}
 \bar{\lambda}(u_n, \rho, s) &= \mathcal{G}(fu_n, g\rho, s) \triangleleft \mathcal{G}(fu_n, hu_n, s) \triangleleft \mathcal{G}(hu_n, j\rho, s) \\
 &\quad * \mathcal{G}(g\rho, j\rho, s) \triangleleft \mathcal{G}(hu_n, g\rho, s) \triangleleft \mathcal{G}(fu_n, j\rho, s) \\
 \hat{\nu}(u_n, \rho, s) &= \mathcal{D}(fu_n, g\rho, s) \oplus \mathcal{D}(fu_n, hu_n, s) \oplus \mathcal{D}(hu_n, j\rho, s) \\
 &\quad \oplus \mathcal{D}(g\rho, j\rho, s) \oplus \mathcal{D}(hu_n, g\rho, s) \oplus \mathcal{P}(fu_n, j\rho, s) \\
 W(u_n, \rho, s) &= \mathcal{P}(fu_n, g\rho, s) \oplus \mathcal{P}(fu_n, hu_n, s) \oplus \mathcal{P}(hu_n, j\rho, s) \\
 &\quad \oplus \mathcal{P}(g\rho, j\rho, s) \oplus \mathcal{P}(hu_n, g\rho, s) \oplus \mathcal{P}(fu_n, j\rho, s)
 \end{aligned}$$

Taking the limit as $n \rightarrow \infty$ in the inequalities above, we have

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \bar{\lambda}(u_n, \rho, s) &= \mathcal{G}(w, w, s) \triangleleft \mathcal{G}(w, w, s) \triangleleft \mathcal{G}(w, w, s) \triangleleft \mathcal{G}(w, w, s) \triangleleft \mathcal{G}(w, w, s) \triangleleft \mathcal{G}(w, w, s) \\
 &= \mathcal{G}(w, w, s) \triangleleft \mathcal{G}(w, w, s) \triangleleft 1 \triangleleft 1 \triangleleft 1 \triangleleft \mathcal{G}(w, w, s) \\
 &= \mathcal{G}(w, w, s) \triangleleft \mathcal{G}(w, w, s) \triangleleft \mathcal{G}(w, w, s) \\
 &= \min\{\mathcal{G}(w, w, s), \mathcal{G}(w, w, s), \mathcal{G}(w, w, s)\} \\
 &= \mathcal{G}(w, w, s) \text{ and} \\
 \lim_{n \rightarrow \infty} \hat{\nu}(u_n, \rho, s) &= \mathcal{D}(w, w, s) \oplus \mathcal{D}(w, w, s) \oplus \mathcal{D}(w, w, s), \oplus \mathcal{D}(w, w, s), \oplus \mathcal{D}(w, w, s), \oplus \mathcal{D}(fw, w, s) \\
 &= \mathcal{D}(w, w, s) \oplus \mathcal{D}(w, w, s) \oplus 0 \oplus 0 \oplus 0 \oplus \mathcal{D}(w, w, s) \\
 &= \mathcal{D}(w, w, s) \oplus \mathcal{D}(w, w, s) \oplus \mathcal{D}(w, w, s) \\
 &= \max\{\mathcal{D}(w, w, s) \oplus \mathcal{D}(w, w, s) \oplus \mathcal{D}(w, w, s)\} \\
 &= \mathcal{D}(w, w, s) \text{ and} \\
 \lim_{n \rightarrow \infty} W(u_n, \rho, s) &= \mathcal{P}(w, w, s) \oplus \mathcal{P}(w, w, s) \oplus \mathcal{P}(w, w, s), \oplus \mathcal{P}(w, w, s), \oplus \mathcal{P}(w, w, s), \oplus \mathcal{P}(fw, w, s) \\
 &= \mathcal{P}(w, w, s) \oplus \mathcal{P}(w, w, s) \oplus 0 \oplus 0 \oplus 0 \oplus \mathcal{P}(w, w, s) \\
 &= \mathcal{P}(w, w, s) \oplus \mathcal{P}(w, w, s) \oplus \mathcal{P}(w, w, s) \\
 &= \max\{\mathcal{P}(w, w, s) \oplus \mathcal{P}(w, w, s) \oplus \mathcal{P}(w, w, s)\} \\
 &= \mathcal{P}(w, w, s)
 \end{aligned}$$

In addition

$$\begin{aligned}
 \int_0^{\mathcal{G}(w, w, s)} \hat{\mu}(s) ds &= \lim_{n \rightarrow \infty} \inf \int_0^{\mathcal{G}(fu_n, g\rho, s)} \hat{\mu}(s) ds \\
 &\geq \lim_{n \rightarrow \infty} \inf \mathcal{U} \left(\int_0^{\bar{\lambda}(u_n, \rho, s)} \hat{\mu}(s) ds \right) \\
 &\geq \mathcal{U} \left(\lim_{n \rightarrow \infty} \inf \int_0^{\bar{\lambda}(u_n, \rho, s)} \hat{\mu}(s) ds \right) \\
 &= \mathcal{U} \left(\int_0^{\mathcal{G}(w, w, s)} \hat{\mu}(s) ds \right) \\
 &> \int_0^{\mathcal{G}(w, w, s)} \hat{\mu}(s) ds
 \end{aligned}$$

and

$$\begin{aligned}
 \int_0^{\mathcal{D}(w,w,s)} \hat{\mu}(s)ds &= \lim_{n \rightarrow \infty} \sup \int_0^{\mathcal{D}(fu_n, g\rho, s)} \hat{\mu}(s)ds \\
 &\leq \lim_{n \rightarrow \infty} \sup \phi \left(\int_0^{\bar{\lambda}(u_n, \rho, s)} \hat{\mu}(s)ds \right) \\
 &\leq \phi \left(\lim_{n \rightarrow \infty} \sup \int_0^{\bar{\lambda}(u_n, \rho, s)} \hat{\mu}(s)ds \right) \\
 &= \phi \left(\int_0^{\mathcal{D}(w,w,s)} \hat{\mu}(s)ds \right) \\
 &< \int_0^{\mathcal{D}(w,w,s)} \hat{\mu}(s)ds
 \end{aligned}$$

and

$$\begin{aligned}
 \int_0^{\mathcal{P}(w,w,s)} \hat{\mu}(s)ds &= \lim_{n \rightarrow \infty} \sup \int_0^{\mathcal{P}(fu_n, g\rho, s)} \hat{\mu}(s)ds \\
 &\leq \lim_{n \rightarrow \infty} \sup \Phi \left(\int_0^{\bar{\lambda}(u_n, \rho, s)} \hat{\mu}(s)ds \right) \\
 &\leq \Phi \left(\lim_{n \rightarrow \infty} \sup \int_0^{\bar{\lambda}(u_n, \rho, s)} \hat{\mu}(s)ds \right) \\
 &= \Phi \left(\int_0^{\mathcal{P}(w,w,s)} \hat{\mu}(s)ds \right) \\
 &< \int_0^{\mathcal{P}(w,w,s)} \hat{\mu}(s)ds
 \end{aligned}$$

which results in a contradiction.

Thus, it follows that $g(\rho) = j(\rho)$. Consequently,

$$g\rho = j\rho = w \quad (5)$$

By (3) and (5), we get

$$fz = hz = g\rho = j\rho = w. \quad (6)$$

Next, we demonstrate that w is a common fixed point of the mappings f, g, h , and j .

Given that the pairs (f, h) and (g, j) are weakly compatible,

$$fz = hz \Rightarrow hfgz = fhz.$$

Now, applying (6), we get

$$fw = hw \quad (7)$$

Also,

$$g\rho = j\rho \Rightarrow jg\rho = gj\rho$$

From (6), we get

$$gw = jw. \quad (8)$$

Next, we assert that $f(w) = w$. Suppose, for the sake of contradiction, that $f(w) \neq w$.

By substituting $x = w$ and $y = \rho$ into the conditions of Theorem 2.1, we get

$$\int_0^{\mathcal{G}(fw, g\rho, s)} \hat{\mu}(s)ds \geq \mathcal{U} \left(\int_0^{\bar{\lambda}(w, \rho, s)} \hat{\mu}(s)ds \right)$$

and

$$\int_0^{\mathcal{D}(fw, g\rho, s)} \hat{\mu}(s) ds \leq \phi \left(\int_0^{\hat{\nu}(w, \rho, s)} \hat{\mu}(s) ds \right)$$

and

$$\int_0^{\mathcal{P}(fw, g\rho, s)} \hat{\mu}(s) ds \leq \Phi \left(\int_0^{W(w, \rho, s)} \hat{\mu}(s) ds \right)$$

$\bar{\lambda}(w, \rho, s) = \mathcal{G}(fw, g\rho, s) \triangleleft \mathcal{G}(fw, hw, s) \triangleleft \mathcal{G}(hw, j\rho, s), \triangleleft \mathcal{G}(g\rho, j\rho, s), \triangleleft \mathcal{G}(hw, g\rho, s), \triangleleft \mathcal{G}(fw, j\rho, s),$
 $\hat{\nu}(w, \rho, s) = \mathcal{D}(fw, g\rho, s) \oplus \mathcal{D}(fw, hw, s) \oplus \mathcal{D}(hw, j\rho, s), \oplus \mathcal{D}(g\rho, j\rho, s), \oplus \mathcal{D}(hw, g\rho, s), \oplus \mathcal{D}(fw, j\rho, s),$
 $W(w, \rho, s) = \mathcal{P}(fw, g\rho, s) \oplus \mathcal{P}(fw, hw, s) \oplus \mathcal{P}(hw, j\rho, s), \oplus \mathcal{P}(g\rho, j\rho, s), \oplus \mathcal{P}(hw, g\rho, s), \oplus \mathcal{P}(fw, j\rho, s).$
 Now,

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \bar{\lambda}(w, \rho, s) &= \mathcal{G}(fw, w, s) \triangleleft \mathcal{G}(fw, w, s) \triangleleft \mathcal{G}(w, w, s), \triangleleft \mathcal{G}(w, w, s), \triangleleft \mathcal{G}(w, w, s), \triangleleft \mathcal{G}(fw, w, s) \\
 &= \mathcal{G}(fw, w, s) \triangleleft \mathcal{G}(fw, w, s) \triangleleft 1 \triangleleft 1 \triangleleft 1 \triangleleft \mathcal{G}(fw, w, s) \\
 &= \mathcal{G}(fw, w, s) \triangleleft \mathcal{G}(fw, w, s) \triangleleft \mathcal{G}(fw, w, s) \\
 &= \min\{\mathcal{G}(fw, w, s), \mathcal{G}(fw, w, s), \mathcal{G}(fw, w, s)\} \\
 &= \mathcal{G}(fw, w, s) \text{ and}
 \end{aligned}$$

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \hat{\nu}(w, \rho, s) &= \mathcal{D}(fw, w, s) \oplus \mathcal{D}(fw, w, s) \oplus \mathcal{D}(w, w, s), \oplus \mathcal{D}(w, w, s), \oplus \mathcal{D}(w, w, s), \oplus \mathcal{D}(fw, w, s) \\
 &= \mathcal{D}(fw, w, s) \oplus \mathcal{D}(fw, w, s) \oplus 0 \oplus 0 \oplus 0 \oplus \mathcal{D}(fw, w, s) \\
 &= \mathcal{D}(fw, w, s) \oplus \mathcal{D}(fw, w, s) \oplus \mathcal{D}(fw, w, s) \\
 &= \max\{\mathcal{D}(fw, w, s) \oplus \mathcal{D}(fw, w, s) \oplus \mathcal{D}(fw, w, s)\} \\
 &= \mathcal{D}(fw, w, s) \text{ and}
 \end{aligned}$$

$$\begin{aligned}
 \lim_{n \rightarrow \infty} W(w, \rho, s) &= \mathcal{P}(fw, w, s) \oplus \mathcal{P}(fw, w, s) \oplus \mathcal{P}(w, w, s), \oplus \mathcal{P}(w, w, s), \oplus \mathcal{P}(w, w, s), \oplus \mathcal{P}(fw, w, s) \\
 &= \mathcal{P}(fw, w, s) \oplus \mathcal{P}(fw, w, s) \oplus 0 \oplus 0 \oplus 0 \oplus \mathcal{P}(fw, w, s) \\
 &= \mathcal{P}(fw, w, s) \oplus \mathcal{P}(fw, w, s) \oplus \mathcal{P}(fw, w, s) \\
 &= \max\{\mathcal{P}(fw, w, s) \oplus \mathcal{P}(fw, w, s) \oplus \mathcal{P}(fw, w, s)\} \\
 &= \mathcal{P}(fw, w, s)
 \end{aligned}$$

Additionally

$$\begin{aligned}
 \int_0^{\mathcal{G}(fw, w, s)} \hat{\mu}(s) ds &= \lim_{n \rightarrow \infty} \inf \int_0^{\mathcal{G}(fw, g\rho, s)} \hat{\mu}(s) ds \\
 &\geq \lim_{n \rightarrow \infty} \inf \mathcal{U} \left(\int_0^{\bar{\lambda}(w, \rho, s)} \hat{\mu}(s) ds \right) \\
 &\geq \mathcal{U} \left(\lim_{n \rightarrow \infty} \inf \int_0^{\bar{\lambda}(w, \rho, s)} \hat{\mu}(s) ds \right) \\
 &= \mathcal{U} \left(\int_0^{\mathcal{G}(fw, w, s)} \hat{\mu}(s) ds \right) \\
 &> \int_0^{\mathcal{G}(fw, w, s)} \hat{\mu}(s) ds
 \end{aligned}$$

and

$$\begin{aligned}
 \int_0^{\mathcal{D}(fw,w,s)} \hat{\mu}(s)ds &= \lim_{n \rightarrow \infty} \sup \int_0^{\mathcal{D}(fw,g\rho,s)} \hat{\mu}(s)ds \\
 &\leq \lim_{n \rightarrow \infty} \sup \phi \left(\int_0^{\bar{\lambda}(w,\rho,s)} \hat{\mu}(s)ds \right) \\
 &\leq \phi \left(\lim_{n \rightarrow \infty} \sup \int_0^{\bar{\lambda}(w,\rho,s)} \hat{\mu}(s)ds \right) \\
 &= \phi \left(\int_0^{\mathcal{D}(fw,w,s)} \hat{\mu}(s)ds \right) \\
 &< \int_0^{\mathcal{D}(fw,w,s)} \hat{\mu}(s)ds
 \end{aligned}$$

and

$$\begin{aligned}
 \int_0^{\mathcal{P}(fw,w,s)} \hat{\mu}(s)ds &= \lim_{n \rightarrow \infty} \sup \int_0^{\mathcal{P}(fw,g\rho,s)} \hat{\mu}(s)ds \\
 &\leq \lim_{n \rightarrow \infty} \sup \Phi \left(\int_0^{\bar{\lambda}(w,\rho,s)} \hat{\mu}(s)ds \right) \\
 &\leq \Phi \left(\lim_{n \rightarrow \infty} \sup \int_0^{\bar{\lambda}(w,\rho,s)} \hat{\mu}(s)ds \right) \\
 &= \Phi \left(\int_0^{\mathcal{P}(fw,w,s)} \hat{\mu}(s)ds \right) \\
 &< \int_0^{\mathcal{P}(fw,w,s)} \hat{\mu}(s)ds
 \end{aligned}$$

This leads to a contradiction. Hence, we conclude that $f(w) = w$. Using equation (7), we deduce

$$fw = hw = w. \quad (9)$$

Similarly, by replacing $x = z$ and $y = w$ in the hypothesis of Theorem 2.1, and applying equations (6) and (8), we derive

$$gw = jw = w. \quad (10)$$

Combining (9) and (20), we get

$$fw = hw = w = gw = jw.$$

Therefore, w serves as a common fixed point for the mappings f, g, h , and j . Next, we examine the uniqueness of this common fixed point. Suppose w_1 and w_2 are both common fixed points of f, g, h , and j . By substituting $x = w_1$ and $y = w_2$ into the conditions of Theorem 2.1, we obtain

$$\int_0^{\mathcal{G}(w_1,w_2,s)} \hat{\mu}(s)ds = \int_0^{\mathcal{G}(fw_1,gw_2,s)} \hat{\mu}(s)ds \geq \mathfrak{U} \left(\int_0^{\bar{\lambda}(w_1,w_2,s)} \hat{\mu}(s)ds \right)$$

and

$$\int_0^{\mathcal{D}(w_1,w_2,s)} \hat{\mu}(s)ds = \int_0^{\mathcal{D}(fw_1,gw_2,s)} \hat{\mu}(s)ds \leq \phi \left(\int_0^{\hat{\nu}(w_1,w_2,s)} \hat{\mu}(s)ds \right)$$

and

$$\int_0^{\mathcal{P}(w_1,w_2,s)} \hat{\mu}(s)ds = \int_0^{\mathcal{P}(fw_1,gw_2,s)} \hat{\mu}(s)ds \leq \Phi \left(\int_0^{W(w_1,w_2,s)} \hat{\mu}(s)ds \right)$$

$$\begin{aligned}
 \bar{\lambda}(w_1, w_2, s) &= \mathcal{G}(fw_1, gw_2, s) \triangleleft \mathcal{G}(fw_1, hw_1, s) \triangleleft \mathcal{G}(hw_1, jw_2, s) \triangleleft \\
 &\quad \mathcal{G}(gw_2, jw_2, s) \triangleleft \mathcal{G}(hw_1, gw_2, s) \triangleleft \mathcal{G}(fw_1, jw_2, s) \\
 &= \mathcal{G}(w_1, w_2, s) \triangleleft \mathcal{G}(w_1, w_1, s) \triangleleft \mathcal{G}(w_1, w_2, s) \triangleleft \\
 &\quad \mathcal{G}(w_2, w_2, s) \triangleleft \mathcal{G}(w_1, w_2, s) \triangleleft \mathcal{G}(w_1, w_2, s) \\
 &= \mathcal{G}(w_1, w_2, s) \triangleleft 1 \triangleleft \mathcal{G}(w_1, w_2, s) \triangleleft 1 \triangleleft \mathcal{G}(w_1, w_2, s) \triangleleft \mathcal{G}(w_1, w_2, s) \\
 &= \mathcal{G}(w_1, w_2, s) \triangleleft \mathcal{G}(w_1, w_2, s) \triangleleft \mathcal{G}(w_1, w_2, s) \triangleleft \mathcal{G}(w_1, w_2, s) \\
 &= \min \{ \mathcal{G}(w_1, w_2, s), \mathcal{G}(w_1, w_2, s), \mathcal{G}(w_1, w_2, s), \mathcal{G}(w_1, w_2, s) \} \\
 &= \mathcal{G}(w_1, w_2, s)
 \end{aligned}$$

and

$$\begin{aligned}
 \hat{\nu}(w_1, w_2, s) &= \mathcal{D}(fw_1, gw_2, s) \oplus \mathcal{D}(fw_1, hw_1, s) \oplus \mathcal{D}(hw_1, jw_2, s) \oplus \\
 &\quad \mathcal{D}(gw_2, jw_2, s) \oplus \mathcal{D}(hw_1, gw_2, s) \oplus \mathcal{D}(fw_1, jw_2, s) \\
 &= \mathcal{D}(w_1, w_2, s) \oplus \mathcal{D}(w_1, w_1, s) \oplus \mathcal{D}(w_1, w_2, s) \oplus \\
 &\quad \mathcal{D}(w_2, w_2, s) \oplus \mathcal{D}(w_1, w_2, s) \oplus \mathcal{D}(w_1, w_2, s) \\
 &= \mathcal{D}(w_1, w_2, s) \oplus 0 \oplus \mathcal{D}(w_1, w_2, s) \oplus 0 \oplus \mathcal{D}(w_1, w_2, s) \oplus \mathcal{D}(w_1, w_2, s) \\
 &= \mathcal{D}(w_1, w_2, s) \oplus \mathcal{D}(w_1, w_2, s) \oplus \mathcal{D}(w_1, w_2, s) \oplus \mathcal{D}(w_1, w_2, s) \\
 &= \max \{ \mathcal{D}(w_1, w_2, s), \mathcal{D}(w_1, w_2, s), \mathcal{D}(w_1, w_2, s), \mathcal{D}(w_1, w_2, s) \} \\
 &= \mathcal{D}(w_1, w_2, s).
 \end{aligned}$$

and

$$\begin{aligned}
 W(w_1, w_2, s) &= \mathcal{P}(fw_1, gw_2, s) \oplus \mathcal{P}(fw_1, hw_1, s) \oplus \mathcal{P}(hw_1, jw_2, s) \oplus \\
 &\quad \mathcal{P}(gw_2, jw_2, s) \oplus \mathcal{P}(hw_1, gw_2, s) \oplus \mathcal{P}(fw_1, jw_2, s) \\
 &= \mathcal{P}(w_1, w_2, s) \oplus \mathcal{P}(w_1, w_1, s) \oplus \mathcal{P}(w_1, w_2, s) \oplus \\
 &\quad \mathcal{P}(w_2, w_2, s) \oplus \mathcal{P}(w_1, w_2, s) \oplus \mathcal{P}(w_1, w_2, s) \\
 &= \mathcal{P}(w_1, w_2, s) \oplus 0 \oplus \mathcal{P}(w_1, w_2, s) \oplus 0 \oplus \mathcal{P}(w_1, w_2, s) \oplus \mathcal{P}(w_1, w_2, s) \\
 &= \mathcal{P}(w_1, w_2, s) \oplus \mathcal{P}(w_1, w_2, s) \oplus \mathcal{P}(w_1, w_2, s) \oplus \mathcal{P}(w_1, w_2, s) \\
 &= \max \{ \mathcal{P}(w_1, w_2, s), \mathcal{P}(w_1, w_2, s), \mathcal{P}(w_1, w_2, s), \mathcal{P}(w_1, w_2, s) \} \\
 &= \mathcal{P}(w_1, w_2, s).
 \end{aligned}$$

Thus,

$$\begin{aligned}
 \int_0^{\mathcal{G}(w_1, w_2, s)} \hat{\mu}(s) ds &\geq \mathcal{U} \left(\int_0^{\mathcal{G}(w_1, w_2, s)} \hat{\mu}(s) ds \right) \\
 &> \int_0^{\mathcal{G}(w_1, w_2, s)} \hat{\mu}(s) ds
 \end{aligned}$$

and

$$\begin{aligned}
 \int_0^{\mathcal{D}(w_1, w_2, s)} \hat{\mu}(s) ds &\leq \phi \left(\int_0^{\mathcal{D}(w_1, w_2, s)} \hat{\mu}(s) ds \right) \\
 &< \int_0^{\mathcal{D}(w_1, w_2, s)} \hat{\mu}(s) ds
 \end{aligned}$$

and

$$\begin{aligned}
 \int_0^{\mathcal{P}(w_1, w_2, s)} \hat{\mu}(s) ds &\leq \Phi \left(\int_0^{\mathcal{P}(w_1, w_2, s)} \hat{\mu}(s) ds \right) \\
 &< \int_0^{\mathcal{P}(w_1, w_2, s)} \hat{\mu}(s) ds
 \end{aligned}$$

which leads to a contradiction.

Therefore, we must have $w_1 = w_2$. Consequently, the mappings f, g, h , and j share a unique common fixed point in X . This concludes the proof. $\square$

Example 2.1. Consider the set $X = [0, 4]$ equipped with a neutrosophic metric $(M, N, \mathcal{P})$ defined by, for all $x, y \in X$ and $t > 0$,

$$\mathcal{G}(x, y, t) = e^{-\frac{|x-y|}{t+1}}, \quad \mathcal{D}(x, y, t) = 1 - e^{-\frac{|x-y|}{t+1}}, \quad \mathcal{P}(x, y, t) = \frac{|x-y|}{2(t+1)}.$$

Let the continuous t -norm $\triangleleft$ be the product $a \triangleleft b = a \cdot b$, and the t -conorm $\triangleright$ be $a \triangleright b = a + b - ab$. Define four self-maps $f, g, h, j : X \rightarrow X$ as

$$f(x) = \begin{cases} 2, & x \leq 2 \\ \frac{1}{3}, & x > 2 \end{cases}, \quad g(x) = \begin{cases} 2, & x \leq 2 \\ \frac{1}{4}, & x > 2 \end{cases},$$

$$h(x) = \begin{cases} 2, & x \leq 2 \\ \frac{1}{5}, & x > 2 \end{cases}, \quad j(x) = \begin{cases} 2, & x \leq 2 \\ \frac{1}{3}, & x > 2 \end{cases}.$$

Consider sequences $\{u_n\} = \frac{2}{n}$ and $\{v_n\} = \frac{2}{n+1}$ in X . Then,

$$\lim_{n \rightarrow \infty} f(u_n) = \lim_{n \rightarrow \infty} g(v_n) = \lim_{n \rightarrow \infty} h(u_n) = \lim_{n \rightarrow \infty} j(v_n) = 2.$$

Let $\hat{\mu}(t) = 3t$, $\mathcal{U}(t) = 2t$, and $\phi(t) = \frac{t}{3}$.

For any $x, y \in [0, 2]$, observe

$$f(x) = g(y) = h(x) = j(y) = 2,$$

and thus

$$\mathcal{G}(fx, gy, t) = \mathcal{G}(fx, hx, t) = \mathcal{G}(hx, jy, t) = \mathcal{G}(gy, jy, t) = e^0 = 1,$$

$$\mathcal{D}(fx, gy, t) = \mathcal{D}(fx, hx, t) = \mathcal{D}(hx, jy, t) = \mathcal{D}(gy, jy, t) = 0,$$

and

$$\mathcal{P}(fx, gy, t) = \mathcal{P}(fx, hx, t) = \mathcal{P}(hx, jy, t) = \mathcal{P}(gy, jy, t) = 0.$$

Setting $\bar{\lambda}(x, y, t) = 1$ and $\hat{\nu}(x, y, t) = 0$, and noting

$$\mathcal{U}\left(\int_0^1 \hat{\mu}(s) ds\right) = \mathcal{U}(1.5) = 3 > \int_0^1 \hat{\mu}(s) ds = 1.5,$$

the conditions

$$\int_0^{\mathcal{G}(fx, gy, t)} \hat{\mu}(s) ds \geq \mathcal{U}\left(\int_0^{\bar{\lambda}(x, y, t)} \hat{\mu}(s) ds\right)$$

and

$$\int_0^{\mathcal{D}(fx, gy, t)} \hat{\mu}(s) ds \leq \phi\left(\int_0^{\hat{\nu}(x, y, t)} \hat{\mu}(s) ds\right)$$

and

$$\int_0^{\mathcal{P}(fx, gy, t)} \hat{\mu}(s) ds \leq \Phi\left(\int_0^{W(x, y, t)} \hat{\mu}(s) ds\right)$$

are satisfied.

Hence, by the corresponding fixed-point theorem in neutrosophic metric spaces 2.1, the mappings f, g, h, j share a unique common fixed point in X , which is $x = 2$.

Corollary 2.1. Consider an neutrosophic metric space $(X, \mathcal{G}, \mathcal{D}, \mathcal{P}, \triangleleft, \triangleright)$, where the t -norm and t -conorm are defined by $a \triangleleft b = \min\{a, b\}$ and $a \oplus b = \max\{a, b\}$, respectively. Let f, h , and j be self-maps on X . For all $x, y \in X$ and $t > 0$, we have

$$\int_0^{\mathcal{G}(fx, fy, s)} \hat{\mu}(s) ds \geq \mathcal{U} \left(\int_0^{\bar{\lambda}(x, y, s)} \hat{\mu}(s) ds \right)$$

and

$$\int_0^{\mathcal{D}(fx, fy, s)} \hat{\mu}(s) ds \leq \phi \left(\int_0^{\hat{\nu}(x, y, s)} \hat{\mu}(s) ds \right)$$

and

$$\int_0^{\mathcal{P}(fx, fy, s)} \hat{\mu}(s) ds \leq \Phi \left(\int_0^{W(x, y, s)} \hat{\mu}(s) ds \right)$$

where $\hat{\mu} : \mathbb{R}^+ \rightarrow \mathbb{R}$ is a Lebesgue integrable function that is both summable and non-negative, $\mathcal{U} = \{\mathcal{U} \mid \mathcal{U} : \mathbb{R} \rightarrow \mathbb{R} \text{ is upper semi-continuous, } \mathcal{U}(0) = 0, \text{ and } \mathcal{U}(s) < s \text{ for all } s > 0\}$, and $\phi, \Phi = \{\phi, \Phi \mid \phi, \Phi : \mathbb{R} \rightarrow \mathbb{R}\}$ are comparison functions that are left-continuous, non-decreasing, and satisfy $\phi(s), \Phi(s), \phi(s) > s, \Phi(s) > s$ for all $s > 0$, $\bar{\lambda}(x, y, s) = \mathcal{G}(fx, fy, s) \triangleleft \mathcal{G}(fx, hx, s) \triangleleft \mathcal{G}(hx, jy, s), \triangleleft \mathcal{G}(fy, jy, s), \triangleleft \mathcal{G}(hx, fy, s), \triangleleft \mathcal{G}(fx, jy, s)$ and $\hat{\nu}(x, y, s) = \mathcal{D}(fx, fy, s) \oplus \mathcal{D}(fx, hx, s) \oplus \mathcal{D}(hx, jy, s), \oplus \mathcal{D}(fy, jy, s), \oplus \mathcal{D}(hx, fy, s), \oplus \mathcal{D}(fx, jy, s)$ and $W(x, y, s) = \mathcal{P}(fx, fy, s) \oplus \mathcal{P}(fx, hx, s) \oplus \mathcal{P}(hx, jy, s), \oplus \mathcal{P}(fy, jy, s), \oplus \mathcal{P}(hx, fy, s), \oplus \mathcal{P}(fx, jy, s)$. Additionally, suppose the pairs (f, h) and (f, j) satisfy the common limit range (CLR) property relative to the mappings h and j , respectively. Moreover, if both pairs (f, h) and (f, j) are weakly compatible, then the mappings f, h , and j share a unique common fixed point in X .

Corollary 2.2. Let $(X, \mathcal{G}, \mathcal{D}, \mathcal{P}, \triangleleft, \triangleright)$ be a neutrosophic metric space, where the t -norm and t -conorm are given by $a \triangleleft b = \min\{a, b\}$ and $a \oplus b = \max\{a, b\}$, respectively. Consider two self-maps f and j defined on X , and assume that for all $x, y \in X$ and $t > 0$,

$$\int_0^{\mathcal{G}(fx, fy, s)} \hat{\mu}(s) ds \geq \mathcal{U} \left(\int_0^{\bar{\lambda}(x, y, s)} \hat{\mu}(s) ds \right)$$

and

$$\int_0^{\mathcal{D}(fx, fy, s)} \hat{\mu}(s) ds \leq \phi \left(\int_0^{\hat{\nu}(x, y, s)} \hat{\mu}(s) ds \right)$$

and

$$\int_0^{\mathcal{P}(fx, fy, s)} \hat{\mu}(s) ds \leq \Phi \left(\int_0^{W(x, y, s)} \hat{\mu}(s) ds \right)$$

where $\hat{\mu} : \mathbb{R}^+ \rightarrow \mathbb{R}$ is a non-negative, summable function that is Lebesgue integrable. Let $\mathcal{U} = \{\mathcal{U} \mid \mathcal{U} : \mathbb{R} \rightarrow \mathbb{R} \text{ is upper semi-continuous with } \mathcal{U}(0) = 0 \text{ and } \mathcal{U}(s) < s \text{ for all } s > 0\}$, and $\phi, \Phi = \{\phi, \Phi \mid \phi, \Phi : \mathbb{R} \rightarrow \mathbb{R}\}$ are comparison functions that are left-continuous, non-decreasing, and satisfy $\phi(s), \Phi(s), \phi(s) > s, \Phi(s) > s$ for every $s > 0$.

$\bar{\lambda}(x, y, s) = \mathcal{G}(fx, fy, s) \triangleleft \mathcal{G}(fx, jx, s) \triangleleft \mathcal{G}(jx, jy, s), \triangleleft \mathcal{G}(fy, jy, s), \triangleleft \mathcal{G}(jx, fy, s), \triangleleft \mathcal{G}(fx, jy, s)$ and $\hat{\nu}(x, y, s) = \mathcal{D}(fx, fy, s) \oplus \mathcal{D}(fx, jx, s) \oplus \mathcal{D}(jx, jy, s), \oplus \mathcal{D}(fy, jy, s), \oplus \mathcal{D}(jx, fy, s), \oplus \mathcal{D}(fx, jy, s)$ and $W(x, y, s) = \mathcal{P}(fx, fy, s) \oplus \mathcal{P}(fx, jx, s) \oplus \mathcal{P}(jx, jy, s), \oplus \mathcal{P}(fy, jy, s), \oplus \mathcal{P}(jx, fy, s), \oplus \mathcal{P}(fx, jy, s)$. Assume further that the pair (f, j) satisfies the common limit range (CLR) property with respect to the mapping j . Moreover, if (f, j) is weakly compatible, then the functions f and j share a unique common fixed point in X .

Corollary 2.3. Consider an neutrosophic metric space $(X, \mathcal{G}, \mathcal{D}, \mathcal{P}, \triangleleft, \triangleright)$, where the t -norm is defined by $a \triangleleft b = \min\{a, b\}$ and the t -conorm by $a \oplus b = \max\{a, b\}$. Let f, g, h , and j be self-maps on X , and assume that for all $x, y \in X$ and for any $t > 0$,

$$\int_0^{\mathcal{G}(fx, gy, s)} \hat{\mu}(s) ds \geq \mathcal{U} \left(\int_0^{\eta(x, y, s)} \hat{\mu}(s) ds \right)$$

and

$$\int_0^{\mathcal{D}(fx, gy, s)} \hat{\mu}(s) ds \leq \phi \left(\int_0^{\zeta(x, y, s)} \hat{\mu}(s) ds \right)$$

and

$$\int_0^{\mathcal{P}(fx, gy, s)} \hat{\mu}(s) ds \leq \Phi \left(\int_0^{\epsilon(x, y, s)} \hat{\mu}(s) ds \right)$$

where $\hat{\mu} : \mathbb{R}^+ \rightarrow \mathbb{R}$ is a non-negative, summable function that is Lebesgue integrable. The set $\mathcal{U}$ consists of functions $\mathcal{U} : \mathbb{R} \rightarrow \mathbb{R}$ that are upper semi-continuous, satisfy $\mathcal{U}(0) = 0$, and for which $\mathcal{U}(s) < s$ holds for all $s > 0$.

Similarly, ϕ and Φ denote sets of comparison functions $\phi, \Phi : \mathbb{R} \rightarrow \mathbb{R}$ that are left-continuous, non-decreasing, and satisfy $\phi(s) > s$, and $\Phi(s) > s$ for every $s > 0$.

$\eta(x, y, s) = \mathcal{G}(fx, gy, s) \triangleleft \mathcal{G}(fx, hx, s) \triangleleft \mathcal{G}(gy, jy, s), \triangleleft \mathcal{G}(hx, gy, s), \triangleleft \mathcal{G}(fx, jy, s)$
and $\zeta(x, y, s) = \mathcal{D}(fx, gy, s) \oplus \mathcal{D}(fx, hx, s) \oplus \mathcal{D}(gy, jy, s), \oplus \mathcal{D}(hx, gy, s), \oplus \mathcal{D}(fx, jy, s)$ and $\epsilon(x, y, s) = \mathcal{P}(fx, gy, s) \oplus \mathcal{P}(fx, hx, s) \oplus \mathcal{P}(gy, jy, s), \oplus \mathcal{P}(hx, gy, s), \oplus \mathcal{P}(fx, jy, s)$.

Assume further that the pairs (f, h) and (g, j) satisfy the common limit range (CLR) property relative to the mappings h and j , respectively. Moreover, if both (f, h) and (g, j) are weakly compatible, then the mappings f, g, h , and j admit a unique common fixed point in X .

Corollary 2.4. Let $(X, \mathcal{G}, \mathcal{D}, \mathcal{P}, \triangleleft, \triangleright)$ be an neutrosophic metric space, where the t -norm is defined as $a \triangleleft b = \min\{a, b\}$ and the t -conorm as $a \oplus b = \max\{a, b\}$. Consider four self-mappings f, g, h , and j on X , and suppose that for all $x, y \in X$ and for any $t > 0$,

$$\int_0^{\mathcal{G}(fx, gy, s)} \hat{\mu}(s) ds \geq \mathcal{U} \left(\int_0^{\bar{\lambda}(x, y, s)} \hat{\mu}(s) ds \right)$$

and

$$\int_0^{\mathcal{D}(fx, gy, s)} \hat{\mu}(s) ds \leq \phi \left(\int_0^{\hat{\nu}(x, y, s)} \hat{\mu}(s) ds \right)$$

and

$$\int_0^{\mathcal{P}(fx, gy, s)} \hat{\mu}(s) ds \leq \Phi \left(\int_0^{W(x, y, s)} \hat{\mu}(s) ds \right)$$

where $\hat{\mu} : \mathbb{R}^+ \rightarrow \mathbb{R}$ is a Lebesgue integrable function that is summable and non-negative. The set $\mathcal{U}$ consists of functions $\mathcal{U} : \mathbb{R} \rightarrow \mathbb{R}$ which are upper semi-continuous, satisfy $\mathcal{U}(0) = 0$, and for every $s > 0$, satisfy $\mathcal{U}(s) < s$.

Similarly, ϕ and Φ refer to families of comparison functions $\phi, \Phi : \mathbb{R} \rightarrow \mathbb{R}$ that are left-continuous, non-decreasing, and fulfill $\phi(s) > s$ and $\Phi(s) > s$ for all $s > 0$.

$\bar{\lambda}(x, y, s) = \mathcal{G}(fx, gy, s) \triangleleft \mathcal{G}(fx, hx, s) \triangleleft \mathcal{G}(hx, jy, s), \triangleleft \mathcal{G}(gy, jy, s), \triangleleft \mathcal{G}(hx, gy, s), \triangleleft \mathcal{G}(fx, jy, s)$
and $\hat{\nu}(x, y, s) = \mathcal{D}(fx, gy, s) \oplus \mathcal{D}(fx, hx, s) \oplus \mathcal{D}(hx, jy, s), \oplus \mathcal{D}(gy, jy, s), \oplus \mathcal{D}(hx, gy, s), \oplus \mathcal{D}(fx, jy, s)$
and $W(x, y, s) = \mathcal{P}(fx, gy, s) \oplus \mathcal{P}(fx, hx, s) \oplus \mathcal{P}(hx, jy, s), \oplus \mathcal{P}(gy, jy, s), \oplus \mathcal{P}(hx, gy, s), \oplus \mathcal{P}(fx, jy, s)$.

Assume further that the pairs (f, h) and (g, j) satisfy the (E.A) property, and that either $j(X)$ or $h(X)$ forms a closed subset of X . Moreover, if both pairs (f, h) and (g, j) are weakly compatible, then the mappings f, g, h , and j have a unique common fixed point in X .

3. APPLICATION TO SOLVABILITY OF SYSTEM OF FUNCTIONAL EQUATIONS OCCURING IN DYNAMICAL PROGRAMMING

In this section, we investigate the existence and uniqueness of solutions for a class of functional equations arising in dynamic programming, utilizing Theorem 2.1.

Let A and B be Banach spaces, with $C \subseteq A$ representing the state space and $D \subseteq B$ the decision space.

$$\left\{ \begin{array}{l} S_1 s_1(x) = \text{opt}_{y \in D} \{a(x, y) + A_1(x, y, s_1(\gamma_1(x, y)))\} \\ S_2 s_1(x) = \text{opt}_{y \in D} \{a(x, y) + B_1(x, y, s_1(\gamma_2(x, y)))\} \\ S_3 s_1(x) = \text{opt}_{y \in D} \{b(x, y) + A_2(x, y, s_1(\gamma_3(x, y)))\} \\ S_4 s_1(x) = \text{opt}_{y \in D} \{b(x, y) + B_2(x, y, s_1(\gamma_4(x, y)))\} \\ S_5 s_1(x) = \text{opt}_{y \in D} \{b(x, y) + A_3(x, y, s_1(\gamma_5(x, y)))\} \\ S_6 s_1(x) = \text{opt}_{y \in D} \{b(x, y) + B_3(x, y, s_1(\gamma_6(x, y)))\} \end{array} \right\} \quad (*)$$

where $\gamma_1 : C \times D \rightarrow C$, $a : C \times D \rightarrow \mathbb{R}$, $A_i, B_i : C \times D \times \mathbb{R} \rightarrow \mathbb{R}$ and $s_1 \in E(C)$. Denote by $E(C)$ the space of all real-valued continuous functions defined on C . This space is equipped with a complete metric defined as

$$d(p, q) = \sup_{x \in C} |p(x) - q(x)|$$

for every $p, q \in E(C)$.

As an application of Theorem 2.1, we investigate the existence of solutions to the following system of functional equations arising in dynamic programming.

$$\left\{ \begin{array}{l} S_1(x) = \text{opt}_{y \in D} \{a(x, y) + A_1(x, y, S_1(\gamma_1(x, y)))\} \\ S_2(x) = \text{opt}_{y \in D} \{a(x, y) + A_2(x, y, S_2(\gamma_2(x, y)))\} \\ S_3(x) = \text{opt}_{y \in D} \{b(x, y) + A_3(x, y, S_3(\gamma_3(x, y)))\} \\ S_4(x) = \text{opt}_{y \in D} \{b(x, y) + A_4(x, y, S_4(\gamma_4(x, y)))\} \\ S_5(x) = \text{opt}_{y \in D} \{b(x, y) + A_5(x, y, S_5(\gamma_5(x, y)))\} \\ S_6(x) = \text{opt}_{y \in D} \{b(x, y) + A_6(x, y, S_6(\gamma_6(x, y)))\} \end{array} \right\} \quad (**)$$

Theorem 3.1. Let $A_1, A_2, A_3, B_1, B_2, B_3 : C \times D \times \mathbb{R} \rightarrow \mathbb{R}$ be bounded functions, and let $S_1, S_2, S_3, S_4, S_5, S_6 : E(C) \rightarrow E(C)$ be operators defined as in (*). Assume the following conditions are satisfied:

- (i) The pairs (S_1, S_4) , (S_2, S_3) , and (S_5, S_6) possess the $CLR_{S_4 S_3}$ property.
- (ii) If $S_1 s_1 = S_4 s_1$, then $S_1 S_4 s_1 = S_4 S_1 s_1$; similarly, if $S_2 s_1 = S_3 s_1$, then $S_2 S_3 s_1 = S_3 S_2 s_1$; and if $S_5 s_1 = S_6 s_1$, then $S_5 S_6 s_1 = S_6 S_5 s_1$.
- (iii) The following inequalities hold:

$$\int_0^{|A_1(x, y, s_1(\gamma_1(x, y))) - B_1(x, y, s_2(\gamma_2(x, y)))|} \hat{\mu}(s) ds \geq \mathfrak{U} \left(\int_0^{\theta_1(s_1, s_2, s)} \hat{\mu}(s) ds \right),$$

$$\int_0^{|A_1(x, y, s_1(\gamma_1(x, y))) - B_1(x, y, s_2(\gamma_2(x, y)))|} \hat{\mu}(s) ds \leq \phi \left(\int_0^{\theta_2(s_1, s_2, s)} \hat{\mu}(s) ds \right),$$

and

$$\int_0^{|A_1(x, y, s_1(\gamma_1(x, y))) - B_1(x, y, s_2(\gamma_2(x, y)))|} \hat{\mu}(s) ds \leq \Phi \left(\int_0^{\theta_2(s_1, s_2, s)} \hat{\mu}(s) ds \right).$$

where

$$\theta_1(s_1, s_2, s) = \min \left\{ \frac{s}{s + |S_1x - S_2y|}, \frac{s}{s + |S_1x - S_4x|}, \frac{s}{s + |S_4x - S_3y|}, \right. \\ \left. \frac{s}{s + |S_2y - S_3y|}, \frac{s}{s + |S_4x - S_2y|}, \frac{s}{s + |S_1x - S_3y|} \right\}$$

and

$$\theta_2(s_1, s_2, s) = \max \left\{ \frac{|S_1x - S_2y|}{s + |S_1x - S_2y|}, \frac{|S_1x - S_4x|}{s + |S_1x - S_4x|}, \frac{|S_4x - S_3y|}{s + |S_4x - S_3y|}, \right. \\ \left. \frac{|S_2y - S_3y|}{s + |S_2y - S_3y|}, \frac{|S_4x - S_2y|}{s + |S_4x - S_2y|}, \frac{|S_1x - S_3y|}{s + |S_1x - S_3y|} \right\}$$

and

$$\theta_2(s_1, s_2, s) = \max \left\{ \frac{|S_1x - S_2y|}{s + |S_1x - S_2y|}, \frac{|S_1x - S_4x|}{s + |S_1x - S_4x|}, \frac{|S_4x - S_3y|}{s + |S_4x - S_3y|}, \right. \\ \left. \frac{|S_2y - S_3y|}{s + |S_2y - S_3y|}, \frac{|S_4x - S_2y|}{s + |S_4x - S_2y|}, \frac{|S_1x - S_3y|}{s + |S_1x - S_3y|} \right\}$$

Consequently, the system of functional equations possesses a unique common solution.

Proof. The system of equations admits a unique bounded solution if and only if the operators defined in (*) possess a unique common fixed point. Since the functions $A_1, A_2, A_3, B_1, B_2, B_3 : C \times D \times \mathbb{R} \rightarrow \mathbb{R}$ are bounded, there exists a constant $P \geq 0$ such that

$$\sup \{ |A_i(x, y, z)|, |B_i(x, y, z)| ; (x, y, z) \in C \times D \times \mathbb{R} \} \leq P.$$

Let $\epsilon > 0, x \in C, \gamma_1, \gamma_2 \in E(c)$. Assume that $\text{opt}_{y \in D} = \sup_{y \in D}$. Then there exist $y_1, y_2 \in D$ such that

$$S_1s_1(x) < a(x, y_1) + A_1(x, y_1, s_1(\gamma_1(x, y_1))) + \epsilon \quad (11)$$

$$S_2s_2(x) < a(x, y_2) + B_1(x, y_2, s_2(\gamma_2(x, y_2))) + \epsilon \quad (12)$$

$$S_1s_1(x) \geq a(x, y_2) + A_1(x, y_2, s_1(\gamma_1(x, y_1))) \quad (13)$$

$$S_2s_2(x) \geq a(x, y_1) + B_1(x, y_1, s_2(\gamma_2(x, y_1))) \quad (14)$$

Equations (11) and (14) yields that

$$S_1s_1(x) - S_2s_2(x) < A_1(x, y_1, s_1(\gamma_1(x, y_1))) - B_1(x, y_1, s_2(\gamma_2(x, y_1))) + \epsilon$$

Similarly (12) and (13) yields that

$$S_1s_1(x) - S_2s_2(x) < A_1(x, y_2, s_1(\gamma_1(x, y_2))) - B_1(x, y_2, s_2(\gamma_2(x, y_2))) + \epsilon$$

Therefore,

$$S_1s_1(x) - S_2s_2(x) \leq \max \{ |A_1(x, y_1, s_1(\gamma_1(x, y_1))) - B_1(x, y_1, s_2(\gamma_2(x, y_1)))| \\ |A_1(x, y_2, s_1(\gamma_1(x, y_2))) - B_1(x, y_2, s_2(\gamma_2(x, y_2)))| \} + \epsilon$$

Let $P = A_1(x, y_1, s_1(\gamma_1(x, y_1)))$, $Q = B_1(x, y_1, s_2(\gamma_2(x, y_1)))$, $R = A_1(x, y_2, s_1(\gamma_1(x, y_2)))$ and $S = B_1(x, y_2, s_2(\gamma_2(x, y_2)))$.

At this point, the previous inequality simplifies to

$$S_1s_1(x) - S_2s_2(x) \leq \max\{|P - Q| + \epsilon, |R - S| + \epsilon\} = H(\text{ say }) \quad (15)$$

Applying (15), we get

$$\begin{aligned}
 & \int_0^{S_1 s_1(x) - S_2 s_2(x)} \hat{\mu}(s) ds \leq \int_0^H \hat{\mu}(s) ds \\
 & = \max \left\{ \int_0^{|P-Q|+\epsilon} \hat{\mu}(s) ds, \int_0^{|R-S|+\epsilon} \hat{\mu}(s) ds \right\} \\
 & = \max \left\{ \int_0^{|P-Q|} \hat{\mu}(s) ds + \int_{|P-Q|}^{|P-Q|+\epsilon} \hat{\mu}(s) ds, \int_0^{|R-S|} \hat{\mu}(s) ds + \int_{|R-S|}^{|R-S|+\epsilon} \hat{\mu}(s) ds \right\} \\
 & = \max \left\{ \int_0^{|P-Q|} \hat{\mu}(s) ds, \int_0^{|R-S|} \hat{\mu}(s) ds \right\} + \max \left\{ \int_{|P-Q|}^{|P-Q|+\epsilon} \hat{\mu}(s) ds, \int_{|R-S|}^{|R-S|+\epsilon} \hat{\mu}(s) ds \right\}
 \end{aligned}$$

Applying the third condition from Theorem 3.1, we derive

$$\int_0^{\frac{|S_1 x - S_2 y|}{s + |S_1 x - S_2 y|}} \hat{\mu}(s) ds \leq \phi \left(\int_0^{\theta_2(s_1, s_2, s)} \hat{\mu}(s) ds \right) + \max \left\{ \int_{|P-Q|}^{|P-Q|+\epsilon} \hat{\mu}(s) ds, \int_{|R-S|}^{|R-S|+\epsilon} \hat{\mu}(s) ds \right\}$$

For any $\epsilon > 0$, we can select $\mathcal{G}(L) \leq \epsilon$ so that

$$\int \hat{\mu}(s) ds \leq \delta,$$

where $\mathcal{G}(L)$ denotes the Lebesgue measure.

Now, the above inequality reduces to

$$\int_0^{\frac{|S_1 x - S_2 y|}{s + |S_1 x - S_2 y|}} \hat{\mu}(s) ds \leq \phi \left(\int_0^{\theta_2(s_1, s_2, s)} \hat{\mu}(s) ds \right) + \delta$$

Taking the limit as $\delta \rightarrow 0$, we get

$$\int_0^{\mathcal{D}(S_1 s_1, S_2 s_2, s)} \hat{\mu}(s) ds \leq \phi \left(\int_0^{\theta_2(s_1, s_2, s)} \hat{\mu}(s) ds \right)$$

and

$$\int_0^{\mathcal{P}(S_1 s_1, S_2 s_2, s)} \hat{\mu}(s) ds \leq \Phi \left(\int_0^{\theta_2(s_1, s_2, s)} \hat{\mu}(s) ds \right)$$

Similarly, by applying the third condition of Theorem 3.1, it can be demonstrated that

$$\int_0^{\mathcal{G}(S_1 s_1, S_2 s_2, s)} \hat{\mu}(s) ds \geq \mathcal{U} \left(\int_0^{\theta_2(s_1, s_2, s)} \hat{\mu}(s) ds \right)$$

Therefore, all the hypotheses of Theorem 3.1 are satisfied. Consequently, the system of functional equations (**) admits a unique common solution. $\square$

4. APPLICATION TO FUNCTIONAL EQUATIONS IN DYNAMIC PROGRAMMING

In this section, we apply the main common fixed point theorem to establish the existence and uniqueness of solutions for a class of functional equations arising in dynamic programming.

Let S and D be nonempty sets representing the state space and decision space, respectively. Denote by $\mathcal{B}(S)$ the set of all bounded real-valued functions on S .

Consider the functional equation

$$u(x) = \sup_{y \in D} \{g(x, y) + H(x, y, u(\tau(x, y)))\}, \quad x \in S, \quad (16)$$

where

- (1) $g : S \times D \rightarrow \mathbb{R}$ is a bounded function,
- (2) $\tau : S \times D \rightarrow S$ is a transition mapping,
- (3) $H : S \times D \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous.

Define the operator $T : \mathcal{B}(S) \rightarrow \mathcal{B}(S)$ by

$$(Tu)(x) = \sup_{y \in D} \{g(x, y) + H(x, y, u(\tau(x, y)))\}, \quad x \in S.$$

Then a function $u \in \mathcal{B}(S)$ is a solution of (16) if and only if it is a fixed point of T .

Now define a neutrosophic metric on $\mathcal{B}(S)$ by

$$M(u, v, t) = \frac{t}{t + \|u - v\|_\infty},$$

$$I(u, v, t) = \frac{\|u - v\|_\infty}{t + \|u - v\|_\infty},$$

and

$$F(u, v, t) = \frac{\|u - v\|_\infty}{t + \|u - v\|_\infty}, \quad t > 0,$$

where

$$\|u - v\|_\infty = \sup_{x \in S} |u(x) - v(x)|.$$

Then,

$$(\mathcal{B}(S), M, I, F, *, \diamond)$$

is a complete neutrosophic metric space.

Assume that there exists a constant $k \in (0, 1)$ such that

$$|H(x, y, r) - H(x, y, s)| \leq k|r - s| \quad (17)$$

for all $x \in S$, $y \in D$, and $r, s \in \mathbb{R}$.

Further, define the mappings

$$A = T, \quad B = T^2.$$

Theorem 4.1. *Suppose that condition (17) holds and that the mappings A and B satisfy all the assumptions of the main common fixed point theorem established in this paper. Then the functional equation (16) possesses a unique bounded solution in $\mathcal{B}(S)$.*

Proof. For any $u, v \in \mathcal{B}(S)$ and $x \in S$, we have

$$\begin{aligned} |(Tu)(x) - (Tv)(x)| &= \left| \sup_{y \in D} \{g(x, y) + H(x, y, u(\tau(x, y)))\} \right. \\ &\quad \left. - \sup_{y \in D} \{g(x, y) + H(x, y, v(\tau(x, y)))\} \right| \\ &\leq \sup_{y \in D} |H(x, y, u(\tau(x, y))) - H(x, y, v(\tau(x, y)))| \\ &\leq k \sup_{y \in D} |u(\tau(x, y)) - v(\tau(x, y))| \\ &\leq k \|u - v\|_\infty. \end{aligned}$$

Consequently,

$$\|Tu - Tv\|_\infty \leq k \|u - v\|_\infty.$$

Hence,

$$M(Tu, Tv, t) = \frac{t}{t + \|Tu - Tv\|_\infty} \geq \frac{t}{t + k\|u - v\|_\infty} \geq M(u, v, t),$$

while

$$I(Tu, Tv, t) \leq I(u, v, t),$$

and

$$F(Tu, Tv, t) \leq F(u, v, t).$$

Thus, T satisfies a neutrosophic contractive condition. By the hypotheses, the mappings $A = T$ and $B = T^2$ satisfy the assumptions of the main common fixed point theorem. Therefore, A and B possess a unique common fixed point $u^* \in \mathcal{B}(S)$ such that

$$Au^* = Bu^* = u^*.$$

Since $A = T$, it follows that

$$Tu^* = u^*.$$

Hence, u^* is a solution of (16). The uniqueness of the common fixed point guarantees that this solution is unique. $\square$

5. CONCLUSION

In this paper, we have established new common fixed point results for two pairs of weakly compatible mappings in neutrosophic metric spaces under an integral type contraction condition via the CLR property. These findings extend and unify several existing results in the fixed point theory literature. Moreover, we demonstrated the practical significance of our theoretical results by applying the common fixed point theorem to guarantee the existence and uniqueness of solutions to functional equations arising in dynamic programming for multistage decision processes. An illustrative example was also provided to validate the main results. Future research may explore further generalizations and applications in more complex neutrosophic structures.

FUTURE WORK

Building on the results presented in this paper, several directions for future research are envisioned:

- (1) **Extension to Other Neutrosophic Structures:** Investigate common fixed point theorems in more generalized neutrosophic structures, such as neutrosophic normed spaces or neutrosophic probabilistic metric spaces.
- (2) **Relaxation of Conditions:** Explore fixed point results under weaker compatibility or contractive conditions to broaden the applicability of the theory.
- (3) **Multivalued Mappings:** Extend the current framework to multivalued (set-valued) mappings in neutrosophic metric spaces, which have practical significance in optimization and control theory.
- (4) **Applications to Differential and Integral Equations:** Apply the fixed point theorems to establish existence and uniqueness of solutions to more complex functional, differential, and integral equations modeled within neutrosophic contexts.
- (5) **Algorithm Development:** Design iterative algorithms based on the fixed point results for practical computation of solutions in applied problems, including decision sciences and dynamic systems.
- (6) **Interdisciplinary Applications:** Explore applications in other fields such as machine learning, data analysis under uncertainty, and neutrosophic modeling in engineering and social sciences.

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ROUGH I-STATISTICAL CONVERGENCE OF ORDER α' IN NEUTROSOPHIC NORMED SPACES: CHARACTERIZATIONS AND APPLICATIONS

MUKHTAR AHMAD¹⁾ AND FLORENTIN SMARANDACHE^{2,*)}

¹⁾Algebra Research Group, Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung, Jalan Ganesha no. 10, Bandung, 40132, Indonesia

²⁾University of New Mexico, Mathematics, Physics and Natural Sciences Division, Gallup, NM 87301, USA

*corresponding authors email: smarand@unm.edu

ABSTRACT. This paper investigates rough I-statistical convergence of order α' in neutrosophic normed spaces. We establish the existence of a minimal roughness parameter that guarantees the nonemptiness of the I-statistical limit set and demonstrate that this mode of convergence remains stable under small perturbations. A uniqueness result for the common limit point is derived whenever the intersection of all corresponding limit sets is nonempty. Finally, the concept is characterized in terms of subsequences determined by filter sets and by decomposing sequences into statistically convergent and statistically bounded parts.

Furthermore, the proposed theory is applied to neutrosophic dynamical systems, fixed-point iterations, and sensor-network data fusion. The obtained results establish stability under sparse disturbances, preservation of convergence for perturbed Picard processes, and robust estimation in the presence of sporadic faulty observations. These applications demonstrate the effectiveness of rough I-statistical convergence of order α' in modeling uncertainty, robustness, and long-term behavior in neutrosophic environments.

1. INTRODUCTION

The theory of statistical convergence has become an important area of research due to its wide applicability in summability theory, functional analysis, approximation theory, and dynamical systems. The concept provides a natural generalization of ordinary convergence by allowing exceptional terms to occur on sets of natural density zero. The notion was later extended in various directions through the introduction of ideal convergence by Kostyrko et al. [2] and ideal statistical convergence by Das et al. [3], which offer a flexible framework for studying convergence with respect to admissible ideals.

Another significant generalization is rough convergence, introduced by Phu [4, 5], where convergence is investigated within a prescribed tolerance radius rather than at a single limit point. This concept has attracted considerable attention because it is particularly suitable for analyzing data contaminated by measurement errors, perturbations, and uncertainties. Subsequently, rough statistical convergence was introduced by Aytar [1] and further developed by Maity [6], while rough ideal convergence and its variants were studied by Pal et al. [7], Dündar and Çakan [21], Dündar [20], Dündar and Ulusu [22, 23], Bulut and Or [8], and Ahmad et al. [9, 10]. Related investigations have also been carried out in probabilistic normed spaces, intuitionistic fuzzy normed spaces, and 2-normed spaces [15, 16, 17, 18, 19].

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mukhkh0786@gmail.com¹; 30125701@mahasiswa.unb.ac.id¹; ORCID: 0009-0006-4597-2278
smarand@unm.edu²; ORCID: 0000-0002-5560-5926.

Parallel to these developments, the theory of fuzzy, intuitionistic fuzzy, and neutrosophic structures has emerged as a powerful mathematical tool for modeling uncertainty and indeterminacy. Important contributions to intuitionistic fuzzy metric and normed spaces were made by Park [26], Saadati and Park [27], Lael and Nourouzi [24], Karakuş et al. [25], Mursaleen and Lohani [28], Mursaleen and Mohiuddine [29], and Kumar and Mursaleen [30]. The introduction of neutrosophic theory by Smarandache [11] significantly broadened the scope of uncertainty modeling by incorporating truth, indeterminacy, and falsity components simultaneously. Recently, neutrosophic normed spaces and their convergence structures have been investigated by Kirişçi and Şimşek [12], Khan et al. [13, 14, 31, 32, 33], among others and Recent developments in neutrosophic normed spaces have focused on generalized convergence, summability, and topological structures. Ahmad and Mursaleen [34] introduced deferred statistical summability in neutrosophic (n)-normed linear spaces, while Ahmad and Mursaleen [35] studied completeness and Cartesian products in neutrosophic rectangular (n)-normed spaces. These works provide important foundations for further investigations in neutrosophic functional analysis.

Despite the extensive literature on rough convergence, statistical convergence, ideal convergence, and neutrosophic normed spaces, the study of rough I-statistical convergence of order α' in neutrosophic normed spaces has not yet been systematically developed. The presence of the order parameter α' allows a finer control over the asymptotic behavior of exceptional sets, while the roughness parameter provides a natural mechanism for handling uncertainty and approximation. Consequently, a unified theory combining rough convergence, ideal statistical convergence, and neutrosophic norms is both mathematically meaningful and practically relevant.

Motivated by these observations, in this paper we introduce the notions of rough statistical convergence and rough I-statistical convergence of order α' in neutrosophic normed spaces and investigate their fundamental properties. We establish the existence of a minimal roughness parameter ensuring the nonemptiness of the corresponding rough I-statistical limit set, prove stability under perturbations, and derive a uniqueness theorem for common rough I-statistical limit points. Furthermore, we obtain characterization theorems via filter-generated subsequences and decomposition methods involving I-statistically convergent and I-statistically bounded sequences.

In addition, several applications of the proposed theory are presented. In particular, we investigate rough I-statistical stability in neutrosophic dynamical systems, convergence of perturbed Picard iteration processes in fixed-point theory, and robust data-fusion models in neutrosophic sensor networks. These applications demonstrate that rough I-statistical convergence of order α' provides an effective framework for studying stability, robustness, uncertainty, and long-term behavior in complex neutrosophic environments.

2. NOTATION AND BASIC DEFINITIONS

This section recalls the necessary notation definitions and basic concepts concerning rough ideal statistical convergence and neutrosophic normed linear spaces.

Definition 2.1. [2] *Let $\mathcal{Y} \neq \emptyset$. A family $\mathcal{I} \subset P(\mathcal{Y})$ of subsets of $\mathcal{Y}$ is called an ideal in $\mathcal{Y}$ provided,*

- (i) $\emptyset \in \mathcal{I}$,
- (ii) $R, S \in \mathcal{I} \implies R \cup S \in \mathcal{I}$,
- (iii) For each $R \in \mathcal{I}, S \subset R \implies S \in \mathcal{I}$.

Here, $P(\mathcal{Y})$ denotes the power set of $\mathcal{Y}$, i.e., the set of all its subsets. An ideal $\mathcal{I}$, which is a proper (non-trivial) subset of $P(\mathcal{Y})$ —that is, $\mathcal{I} \neq P(\mathcal{Y})$ —is referred to as admissible if it includes every singleton subset of $\mathcal{Y}$.

Basic Sets and Parameters	
$\mathcal{N}$	Set of all natural numbers.
$\alpha' \in (0, 1]$	Order of statistical and ideal statistical convergence.
$j' \geq 0$	Roughness parameter controlling the allowable deviation in convergence.
$\lambda^* \in (0, 1)$	Neutrosophic precision parameter.
$\varepsilon > 0$	Arbitrarily small positive real number.
Ideals and Filters	
$\mathbf{I}$	Non-trivial admissible ideal on $\mathcal{N}$.
$\mathcal{F}(\mathbf{I})$	Filter associated with the ideal $\mathbf{I}$.
$\delta_{\mathbf{I}}^*(A)$	$\mathbf{I}$ -asymptotic density of order α' for $A \subseteq \mathcal{N}$.
Spaces and Structures	
$\mathcal{Y}$	Linear space over a field F .
$(\mathcal{Y}, \ \cdot\)$	Normed linear space.
$(\mathcal{Y}, \Upsilon, \Omega, \Gamma)$	Neutrosophic normed space (NNS).
Neutrosophic Norm Functions	
$\Upsilon(y, \lambda^*)$	Degree of membership of y .
$\Omega(y, \lambda^*)$	Degree of non-membership of y .
$\Gamma(y, \lambda^*)$	Degree of indeterminacy of y .
Sequences and Convergence	
$y = \{y'_m\}$	A sequence in $\mathcal{Y}$.
$y'_m \xrightarrow{(\Upsilon, \Omega, \Gamma)} \xi$	Convergence in a neutrosophic normed space.
$y'_m \xrightarrow{j'(\Upsilon, \Omega, \Gamma)} \xi$	Rough convergence in a neutrosophic normed space.
$y'_m \xrightarrow{j'-st(\alpha')} \xi$	Rough statistical convergence of order α' .
$y'_m \xrightarrow{j'-\mathbf{I}-st(\alpha')} \xi$	Rough $\mathbf{I}$ -statistical convergence of order α' .
$\text{LIM}_y^{j'}$	Set of all rough limit points of the sequence y .
$\mathbf{I}-st\text{-LIM}_y^{\alpha'} y$	Set of rough $\mathbf{I}$ -statistical limit points of order α' .

Throughout this paper, we assume that $\mathbf{I}$ represents a non-trivial admissible ideal defined on the set of natural numbers.

Definition 2.2. [2] Let $\mathcal{Y} \neq \emptyset$. A non-empty class $\mathcal{F} \subset P(\mathcal{Y})$ is called filter on $\mathcal{Y}$ provided,

- (i) $\emptyset \notin \mathcal{F}$,
- (ii) $S, T \in \mathcal{F} \implies S \cap T \in \mathcal{F}$,
- (iii) For each $S \in \mathcal{F}, S \subset T \implies T \in \mathcal{F}$.

Every ideal $\mathbf{I}$ is associated with a filter $\mathcal{F}(\mathbf{I})$ defined as follows:

$$\mathcal{F}(\mathbf{I}) = \{\mathbb{M} \subseteq \mathcal{Y} : \mathbb{M}^c \in \mathbf{I}\} \text{ where } \mathbb{M}^c = \mathcal{Y} - \mathbb{M}.$$

Definition 2.3. [2] A sequence $y = \{y'_m\}$ in $\mathcal{Y}$ is called ideal convergent ($\mathbf{I}$ – convergent) to η if for every $\varepsilon > 0$

$$\mathcal{A}(\varepsilon) = \{n \in \mathcal{N} : |y'_m - \eta| \geq \varepsilon\} \in \mathbf{I}.$$

Here, η is known as the $\mathbf{I}$ -limit of the sequence $y = \{y'_m\}$.
Now, we recall rough convergence as follows:

Definition 2.4. [4] Let $(\mathcal{Y}, \|\cdot\|)$ be a normed space. A sequence $y = y'_m$ in $\mathcal{Y}$ is said to be roughly convergent (or j' -convergent) to a point $\xi_1 \in \mathcal{Y}$ with roughness level $j' \geq 0$ if, for every $\varepsilon > 0$, there exists an index $m_0 \in \mathcal{N}$ such that the inequality $|y'_m - \xi_1| < j' + \varepsilon$ holds for all $m \geq m_0$.

This type of convergence is denoted by $y'_m \xrightarrow{r} \xi_1$ or alternatively as $r\text{-}\lim_{m \rightarrow \infty} y'_m = \xi_1$, where the constant j' is referred to as the roughness degree associated with the sequence $y = \{y'_m\}$.

For a sequence $y = \{y'_m\}$ in the normed space $\mathcal{Y}$, the j' -limit set is defined as

$$\text{LIM}_{y'_m}^r = \{\xi_1 \in \mathbb{X} : y'_m \xrightarrow{r} \xi_1\}.$$

In the case where $y = \{y'_m\}$ is a sequence of real numbers, the j' -limit set can also be expressed as

$$\text{LIM}_{y'_m}^r = [\liminf y'_m - j', \limsup y'_m + j'] \quad (\text{see [4]}).$$

Definition 2.5. [7] A sequence $y = y'_m$ is said to be roughly ideal convergent (denoted as $r\text{-I-convergent}$) to a point ξ_1 , where $j' \geq 0$ represents the roughness level, if for every $\varepsilon > 0$,

$$\{m \in \mathcal{N} : \|y'_m - \xi_1\| \geq j' + \varepsilon\} \in \mathcal{I}.$$

Next, we discuss INNLS and the convergence of sequences within this space.

Definition 2.6. The three-tuple structure $(\mathcal{Y}, \Upsilon, \Omega, \Gamma)$ be an NNS, where $\mathcal{Y}$ is a linear space over a field F . Υ, Ω, Γ are called neutrosophic normed space (NNS) on $\mathcal{Y} \times (0, \infty)$ and represent the degree of membership and non-membership on $\mathcal{Y} \times (0, 1)$ if the following conditions hold, for every $y, w \in \mathcal{Y}$ and $\lambda_1^*, \lambda_2^* > 0$:

- (1) $\Upsilon(y, \lambda^*) + \Omega(y, \lambda^*) \leq 1$.
- (2) $\Upsilon(y, \lambda^*) > 0$.
- (3) $\Upsilon(y, \lambda^*) = 1 \Leftrightarrow y = 0$.
- (4) $\Upsilon(cy, \lambda^*) = \Upsilon\left(y, \frac{\lambda^*}{|c|}\right)$, if $c \neq 0, c \in F$.
- (5) $\Upsilon(y, \lambda_1^*) \star \Upsilon(w, \lambda_2^*) \leq \Upsilon(y + w, \lambda_1^* + \lambda_2^*)$.
- (6) $\Upsilon(y, \cdot) : (0, \infty) \rightarrow [0, 1]$ is continuous.
- (7) $\lim_{\lambda^* \rightarrow \infty} \Upsilon(y, \lambda^*) = 1, \lim_{\lambda^* \rightarrow 0} \Upsilon(y, \lambda^*) = 0$.
- (8) $\Omega(y, \lambda^*) < 1$.
- (9) $\Omega(y, \lambda^*) = 0 \Leftrightarrow y = 0$.
- (10) $\Omega(cy, \lambda^*) = \Omega\left(y, \frac{\lambda^*}{|c|}\right)$ if $c \neq 0, c \in F$.
- (11) $\Omega(y, \lambda_1^*) \diamond \Omega(w, \lambda_2^*) \geq \Omega(y + w, \lambda_1^* + \lambda_2^*)$.
- (12) $\Omega(y, \cdot) : (0, \infty) \rightarrow [0, 1]$ is continuous.
- (13) $\lim_{\lambda^* \rightarrow \infty} \Omega(y, \lambda^*) = 0, \lim_{\lambda^* \rightarrow 0} \Omega(y, \lambda^*) = 1$.
- (14) $\Gamma(y, \lambda^*) = 0 \Leftrightarrow y = 0$.
- (15) $\Gamma(cy, \lambda^*) = \Gamma\left(y, \frac{\lambda^*}{|c|}\right)$ if $c \neq 0, c \in F$.
- (16) $\Gamma(y, \lambda_1^*) \diamond \Gamma(w, \lambda_2^*) \geq \Gamma(y + w, \lambda_1^* + \lambda_2^*)$.
- (17) $\Gamma(y, \cdot) : (0, \infty) \rightarrow [0, 1]$ is continuous.
- (18) $\lim_{\lambda^* \rightarrow \infty} \Gamma(y, \lambda^*) = 0, \lim_{\lambda^* \rightarrow 0} \Gamma(y, \lambda^*) = 1$.

Then Υ, Ω, Γ are called neutrosophic normed (NN).

Example 2.1. Let $\mathcal{Y} = \ell^\infty = \left\{y = (y_k)_{k \geq 1} : \|y\|_\infty = \sup_{k \geq 1} |y_k| < \infty\right\}$, be the Banach space of all bounded real sequences equipped with the supremum norm.

Define

$$a \star b = ab, \quad a \diamond b = \max\{a, b\},$$

for all $a, b \in [0, 1]$.

For every $y \in \mathcal{Y}$ and $\lambda^* > 0$, define

$$\begin{aligned}\Upsilon(y, \lambda^*) &= \frac{\lambda^*}{\lambda^* + \|y\|_\infty}, \\ \Omega(y, \lambda^*) &= \frac{\|y\|_\infty}{2(\lambda^* + \|y\|_\infty)}, \\ \Gamma(y, \lambda^*) &= \frac{\|y\|_\infty}{2(\lambda^* + \|y\|_\infty)}.\end{aligned}$$

We will show that $(\mathcal{Y}, \Upsilon, \Omega, \Gamma)$ is a neutrosophic normed space.

First,

$$\Upsilon(y, \lambda^*) + \Omega(y, \lambda^*) = \frac{\lambda^* + \frac{1}{2}\|y\|_\infty}{\lambda^* + \|y\|_\infty} \leq 1,$$

which verifies condition (1).

Moreover,

$$\Upsilon(y, \lambda^*) > 0$$

for every $y \in \mathcal{Y}$, and

$$\Upsilon(y, \lambda^*) = 1 \iff \|y\|_\infty = 0 \iff y = 0,$$

so conditions (2) and (3) hold.

For any nonzero scalar c ,

$$\Upsilon(cy, \lambda^*) = \frac{\lambda^*}{\lambda^* + |c|\|y\|_\infty} = \Upsilon\left(y, \frac{\lambda^*}{|c|}\right),$$

establishing condition (4). Using

$$\|y + w\|_\infty \leq \|y\|_\infty + \|w\|_\infty,$$

we obtain

$$\Upsilon(y, \lambda_1^*) \star \Upsilon(w, \lambda_2^*) = \frac{\lambda_1^*}{\lambda_1^* + \|y\|_\infty} \cdot \frac{\lambda_2^*}{\lambda_2^* + \|w\|_\infty} \leq \frac{\lambda_1^* + \lambda_2^*}{\lambda_1^* + \lambda_2^* + \|y + w\|_\infty} = \Upsilon(y + w, \lambda_1^* + \lambda_2^*),$$

which verifies (5). The continuity of Υ follows immediately from its definition. Furthermore,

$$\lim_{\lambda^* \rightarrow \infty} \Upsilon(y, \lambda^*) = 1, \quad \lim_{\lambda^* \rightarrow 0^+} \Upsilon(y, \lambda^*) = 0,$$

for every $y \neq 0$, proving (6) and (7). Similarly,

$$0 \leq \Omega(y, \lambda^*) < 1,$$

and

$$\Omega(y, \lambda^*) = 0 \iff \|y\|_\infty = 0 \iff y = 0,$$

which establishes (8) and (9). The homogeneity property

$$\Omega(cy, \lambda^*) = \Omega\left(y, \frac{\lambda^*}{|c|}\right)$$

follows directly from the norm structure, yielding (10). Using the triangle inequality,

$$\Omega(y + w, \lambda_1^* + \lambda_2^*) = \frac{\|y + w\|_\infty}{2(\lambda_1^* + \lambda_2^* + \|y + w\|_\infty)} \leq \max\{\Omega(y, \lambda_1^*), \Omega(w, \lambda_2^*)\},$$

which proves (11). Conditions (12) and (13) are immediate since

$$\lim_{\lambda^* \rightarrow \infty} \Omega(y, \lambda^*) = 0, \quad \lim_{\lambda^* \rightarrow 0^+} \Omega(y, \lambda^*) = 1 \quad (y \neq 0).$$

Since

$$\Gamma(y, \lambda^*) \stackrel{95}{=} \Omega(y, \lambda^*),$$

conditions (14)–(18) follow exactly as above.

Hence, all axioms of Definition 2.6 are satisfied. Therefore, $(\ell^\infty, \Upsilon, \Omega, \Gamma)$ is a neutrosophic normed space.

Example 2.2. Let $(\mathcal{Y}, \|\cdot\|)$ be any real normed space. For every $q > 0$ and all $y \in \mathcal{Y}$, define $\Upsilon(y, q) = \frac{q}{q + \|y\|}$, $\Omega(y, q) = \frac{y}{q + \|y\|}$, and $\Gamma(y, q) = \frac{y}{q + \|y\|}$. Then, $(\mathcal{Y}, \Upsilon, \Omega, \Gamma)$ is an INNLS.

Definition 2.7. Let $(\mathcal{Y}, \Upsilon, \Omega, \Gamma)$ be an NNS with neutrosophic norms $(\Upsilon, \Omega, \Gamma)$. A sequence $y = \{y'_m\}$ in Y is called convergent to $\xi_1 \in Y$ with respect to the norm $(\Upsilon, \Omega, \Gamma)$ if for every $\varepsilon > 0$ and $\lambda^* \in (0, 1)$ there exists $m_0 \in \mathcal{N}$ such that $\Upsilon(y'_m - \xi_1; \varepsilon) > 1 - \lambda^*$, and $\Omega(y'_m - \xi_1; \varepsilon) < \lambda^*$, and $\Gamma(y'_m - \xi_1; \varepsilon) < \lambda^*$ for all $m \geq m_0$. It is denoted by $(\Upsilon, \Omega, \Gamma) - \lim_{m \rightarrow \infty} y'_m = \xi_1$.

Definition 2.8. Consider $(\mathcal{Y}, \Upsilon, \Omega, \Gamma)$ as an neutrosophic normed space. A sequence $y = \{y'_m\}$ in $\mathcal{Y}$ is considered rough convergent to $\xi'_1 \in \mathcal{Y}$ concerning the norm $(\Upsilon, \Omega, \Gamma)$ for some non-negative real number j' if there exists an $m_0 \in \mathcal{N}$ such that for every $\varepsilon > 0$ and $\lambda^* \in (0, 1)$, the following holds

$$\Upsilon(y'_m - \xi_1; j' + \varepsilon) > 1 - \lambda^* \text{ and } \Omega(y'_m - \xi_1; j' + \varepsilon) < \lambda^* \text{ and } \Gamma(y'_m - \xi_1; j' + \varepsilon) < \lambda^* \text{ for all } m \geq m_0.$$

It is denoted by $y'_m \xrightarrow{j'(\Upsilon, \Omega, \Gamma)} \xi_1$ or $j'(\Upsilon, \Omega, \Gamma) - \lim_{m \rightarrow \infty} y'_m = \xi_1$.

Remark 2.1. For $r = 0$, the rough convergence agrees with the usual convergence for the sequences in an NNS.

Definition 2.9. Consider $(\mathcal{Y}, \Upsilon, \Omega, \Gamma)$ as a neutrosophic normed space. A sequence $y = \{y'_m\}$ in $\mathcal{Y}$ is considered to be roughly statistically convergent to $\xi_1 \in \mathcal{Y}$ concerning the norm $(\Upsilon, \Omega, \Gamma)$ for a non-negative number j' if for all $\varepsilon > 0$ and $\lambda^* \in (0, 1)$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \{m \leq n : \Upsilon(y'_m - \xi_1; j' + \varepsilon) \leq 1 - \lambda^* \text{ or } \Omega(y'_m - \xi_1; j' + \varepsilon) \geq \lambda^* \text{ or } \Gamma(y'_m - \xi_1; j' + \varepsilon) \geq \lambda^*\} = 0$$

It is represented as $y'_m \xrightarrow{j'-st} \xi_1$ or $j' - st - \lim_{m \rightarrow \infty} y'_m = \xi_1$.

Let $st - LIM_y^{j'}$ represent the collection of all rough statistical limit points of the sequence $y = \{y'_m\}$.

3. MAIN RESULTS

In this section, we formulate the concepts of rough statistical convergence and rough ideal statistical convergence of order $\alpha' \in (0, 1]$ in the framework of neutrosophic normed spaces. We then develop several fundamental properties associated with these convergence notions. Throughout the paper, the order parameter α' is taken from the interval $(0, 1]$.

Definition 3.1. Let $(\mathcal{Y}, \Upsilon, \Omega, \Gamma)$ be an neutrosophic normed space. A sequence $y = \{y'_m\}$ in Y is said to be rough statistically convergent of order α' to $\xi_1 \in \mathcal{Y}$ with respect to norm $(\Upsilon, \Omega, \Gamma)$ for some non-negative number j' if for every $\varepsilon > 0$ and $\lambda^* \in (0, 1)$.

$$\lim_{n \rightarrow \infty} \frac{1}{n^{\alpha'}} \{m \leq n : \Upsilon(y'_m - \xi_1; j' + \varepsilon) \leq 1 - \lambda^* \text{ or } \Omega(y'_m - \xi_1; j' + \varepsilon) \geq \lambda^* \text{ or } \Gamma(y'_m - \xi_1; j' + \varepsilon) \geq \lambda^*\} = 0.$$

It is denoted by $y'_m \xrightarrow{j'-st(\alpha')} \xi_1$ or $j' - st(\alpha') - \lim_{m \rightarrow \infty} y'_m = \xi_1$.

Let $st - LIM_r^{\alpha'} y$ denotes the the set of all rough statistical limit points of order α' of the sequence $y = \{y'_m\}$.

Remark 3.1. For $r = 0$, the notion rough statistical convergence of order α' agrees with the statistical convergence of order α' for the sequences on an NNS and for $\alpha' = 1$, this notion coincides with rough statistical convergence.

Definition 3.2. Let $(\mathcal{Y}, \Upsilon, \Omega, \Gamma)$ be an neutrosophics normed space. A sequence $y = \{y'_m\}$ in $\mathcal{Y}$ is said to be rough ideal statistically convergent of order α' to $\xi_1 \in \mathcal{Y}$ with respect to norm $(\Upsilon, \Omega, \Gamma)$ for some non-negative number j' if for every $\varepsilon > 0$ and $\lambda^* \in (0, 1)$.

$$\left\{ n \in \mathcal{N} : \frac{1}{n^{\alpha'}} \left| \left\{ m \leq n : \begin{array}{l} \Upsilon(y'_m - \xi_1; j' + \varepsilon) \leq 1 - \lambda^* \\ \text{or } \Omega(y'_m - \xi_1; j' + \varepsilon) \geq \lambda^* \\ \text{or } \Gamma(y'_m - \xi_1; j' + \varepsilon) \geq \lambda^* \end{array} \right\} \right| \geq \delta^* \right\} \in \mathbf{I},$$

or

$$\delta_I^* (\{m \in \mathcal{N} : \Upsilon(y'_m - \xi_1; j' + \varepsilon) \leq 1 - \lambda^* \text{ or } \Omega(y'_m - \xi_1; j' + \varepsilon) \geq \lambda^* \text{ or } \Gamma(y'_m - \xi_1; j' + \varepsilon) \geq \lambda^*\}) = 0,$$

where $\delta_I^*(A) = I - \lim_{n \rightarrow \infty} \frac{1}{n^{\alpha'}} |\{m \leq n : m \in A\}|$ if exists.

It is denoted by $y'_m \xrightarrow{j' - \bar{I} - st(\alpha')} \xi_1$ or $j' - st_I^{\alpha'} - \lim_{m \rightarrow \infty} y'_m = \xi_1$.

Let $\mathbf{I} - st - \text{LIM}_{j'}^{\alpha'} y$ denotes the the set of all rough ideal statistical limit points of order α' of the sequence $y = \{y'_m\}$.

Remark 3.2. For $r = 0$, the notion rough ideal statistical convergence of order α' agrees with the ideal statistical convergence of order α' in an NNS.

The following example shows that there is a sequence which is neither rough ideal convergent nor ideal statistical convergent but it is rough ideal statistical convergent of order α' .

Example 3.1. Let S be an infinite subset of $\mathcal{N}$ and $\mathbf{I}$ be an ideal such that $S \in I$. Then $S^c = \{a_1 < a_2 < a_3 < \dots\} \in \mathcal{F}(\mathbf{I})$. Consider a sequence $y = \{y'_m\}_{m \in \mathcal{N}}$ such that

$$y'_m = \begin{cases} (-1)^m, & \text{if } m \notin S \text{ and } \alpha' = 1, \\ m, & \text{if } m \in S. \end{cases}$$

Then

$$\mathbf{I} - st - \text{LIM}_{j'}^{\alpha'} y = \begin{cases} \phi & j' < 1, \\ [1 - j', j' - 1] & j' \geq 0. \end{cases}$$

In general, a sequence's rough limit might not be unique. Therefore, we consider the rough $\mathbf{I}$ -st-limit set of order α' of sequences $y = \{y'_m\}$ with respect to the norm $(\Upsilon, \Omega, \Gamma)$ as

$$\mathbf{I} - st - \text{LIM}_{j'}^{\alpha'} y = \left\{ \xi_1^* \in Y : y'_m \xrightarrow{j' - \mathbf{I} - st(\alpha')} \xi_1^* \right\}.$$

Moreover, sequence $y = \{y'_m\}$ is rough $\mathbf{I}$ -st-convergent provided $\mathbf{I} - st - \text{LIM}_{j'}^{\alpha'} y \neq \phi$ for some $j' > 0$. In [29], it was observed that for a sequence $y = \{y'_m\}$ of real numbers, the set of rough limit points,

$$\text{LIM}_y^{j'} = [\limsup y - j', \liminf y + j'].$$

In the similar way,

$$\mathbf{I} - st - \text{LIM}_{j'}^{\alpha'} y = [I - st - \limsup y - j', I - st - \liminf y + j'].$$

Definition 3.3. A sequence $y = \{y'_m\}$ in neutrosophic normed space $(\mathcal{Y}, \Upsilon, \Omega, \Gamma)$ is $\mathbf{I}$ -st-bounded of order α' (where $\alpha' \in (0, 1]$) if for $\varepsilon > 0$, $\lambda^* \in (0, 1)$ and some $j' > 0$, there exists $\mathcal{H} > 0$ such that

$$\left\{ n \in \mathbb{N} : \frac{1}{n^{\alpha'}} \{m \leq n : \Upsilon(y'_m; \mathcal{H}) \leq 1 - \lambda^* \text{ or } \Omega(y'_m; \mathcal{H}) \geq \lambda^* \text{ or } \Gamma(y'_m; \mathcal{H}) \geq \lambda^*\} \geq \delta^* \right\} \in \mathbf{I}$$

Example 3.2. Consider the neutrosophic normed space $(\mathbb{R}, \Upsilon, \Omega, \Gamma)$, where

$$\Upsilon(x; t) = \frac{t}{t + |x|}, \quad \Omega(x; t) = \frac{|x|}{2(t + |x|)}, \quad \Gamma(x; t) = \frac{|x|}{2(t + |x|)},$$

for every $x \in \mathbb{R}$ and $t > 0$.

Let $\mathbf{I} = \mathbf{I}_f$ be the ideal of all finite subsets of $\mathbb{N}$, and define the sequence

$$y'_m = \begin{cases} m, & \text{if } m = k^2 \text{ for some } k \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

Now, we show that $\{y'_m\}$ is $\mathbf{I}$ -st-bounded of order $\alpha' = 1$.

Fix $\lambda^* \in (0, 1)$ and choose $\mathcal{H} > 0$. Observe that for all non-square integers m ,

$$y'_m = 0,$$

and therefore

$$\Upsilon(y'_m; \mathcal{H}) = 1, \quad \Omega(y'_m; \mathcal{H}) = 0, \quad \Gamma(y'_m; \mathcal{H}) = 0.$$

Hence, non-square indices never belong to the set

$$A_n = \left\{ m \leq n : \Upsilon(y'_m; \mathcal{H}) \leq 1 - \lambda^* \text{ or } \Omega(y'_m; \mathcal{H}) \geq \lambda^* \text{ or } \Gamma(y'_m; \mathcal{H}) \geq \lambda^* \right\}.$$

The only possible elements of A_n are square numbers not exceeding n . Since the number of squares less than or equal to n is $\lfloor \sqrt{n} \rfloor$, we obtain

$$|A_n| \leq \sqrt{n}.$$

Consequently,

$$\frac{|A_n|}{n} \leq \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}} \rightarrow 0 \quad (n \rightarrow \infty).$$

Let $\delta^* > 0$ be arbitrary. Then, there exists $n_0 \in \mathbb{N}$ such that

$$\frac{|A_n|}{n} < \delta^* \quad \text{for all } n \geq n_0.$$

Therefore,

$$\left\{ n \in \mathbb{N} : \frac{1}{n} |A_n| \geq \delta^* \right\} \subseteq \{1, 2, \dots, n_0 - 1\},$$

which is a finite set and hence belongs to $\mathbf{I}_f$.

Thus, $\{y'_m\}$ is $\mathbf{I}$ -statistically bounded of order $\alpha' = 1$.

Notice that the sequence is not bounded in the ordinary sense since $y'_{k^2} = k^2 \rightarrow \infty$. Nevertheless, the unbounded terms occur only on a set of natural density zero, which guarantees $\mathbf{I}$ -statistical boundedness.

We first establish the existence of a minimal roughness parameter for which the $\mathbf{I}$ -statistical limit set of order α' is nonempty.

Theorem 3.1. Let $\{y'_m\}$ be a sequence in a neutrosophic normed space $(\mathcal{Y}, \Upsilon, \Omega, \Gamma)$ such that

$$\mathbf{I}\text{-st-LIM}_{j'}^{\alpha'}(y) \neq \emptyset \quad \text{for some } j' > 0.$$

Define

$$j_0 = \inf \{j' > 0 : \mathbf{I}\text{-st-LIM}_{j'}^{\alpha'}(y) \neq \emptyset\}.$$

Then

$$\text{I-st-LIM}_{j_0}^{\alpha'}(y) \neq \emptyset.$$

Proof. Since $\text{I-st-LIM}_{j'}^{\alpha'}(y) \neq \emptyset$ for some $j' > 0$, there exists $\xi_1 \in \mathcal{Y}$ such that for every $\varepsilon > 0$ and $\lambda^* \in (0, 1)$,

$$\left\{ n \in \mathcal{N} : \frac{1}{n^{\alpha'}} \left\{ m \leq n : \Upsilon(y'_m - \xi_1; j' + \varepsilon) \leq 1 - \lambda^* \right. \right. \\ \left. \text{or } \Omega(y'_m - \xi_1; j' + \varepsilon) \geq \lambda^* \right. \\ \left. \left. \text{or } \Gamma(y'_m - \xi_1; j' + \varepsilon) \geq \lambda^* \right\} \geq \delta^* \right\} \in \mathbf{I}.$$

Hence, the set $\mathcal{J} = \{j' > 0 : \text{I-st-LIM}_{j'}^{\alpha'}(y) \neq \emptyset\}$ is nonempty and bounded below by 0. Therefore, $j_0 = \inf \mathcal{J}$ exists.

By definition of infimum, there exists a decreasing sequence $\{j_k\} \subset \mathcal{J}$ such that

$$j_k > j_0 \quad \text{for all } k, \quad \text{and} \quad j_k \downarrow j_0.$$

For each k , choose

$$\xi_k \in \text{I-st-LIM}_{j_k}^{\alpha'}(y).$$

Then for every $\varepsilon > 0$ and $\lambda^* \in (0, 1)$. For each $k \in \mathbb{N}$, define a set

$$E_k = \left\{ n \in \mathcal{N} : \frac{1}{n^{\alpha'}} \left\{ m \leq n : \Upsilon(y'_m - \xi_k; j_k + \varepsilon) \leq 1 - \lambda^* \right. \right. \\ \left. \text{or } \Omega(y'_m - \xi_k; j_k + \varepsilon) \geq \lambda^* \right. \\ \left. \left. \text{or } \Gamma(y'_m - \xi_k; j_k + \varepsilon) \geq \lambda^* \right\} \geq \delta^* \right\} \in \mathbf{I}.$$

Similarly, since $\{y'_m\}$ is I-st-bounded of order α' , there exists $\mathcal{H} > 0$ such that

$$E_0 = \left\{ n \in \mathcal{N} : \frac{1}{n^{\alpha'}} \left\{ m \leq n : \Upsilon(y'_m; \mathcal{H}) \leq 1 - \lambda^* \right. \right. \\ \left. \text{or } \Omega(y'_m; \mathcal{H}) \geq \lambda^* \right. \\ \left. \left. \text{or } \Gamma(y'_m; \mathcal{H}) \geq \lambda^* \right\} \geq \delta^* \right\} \in \mathbf{I}.$$

Since $\mathbf{I}$ is admissible, the set

$$\mathcal{N} \setminus (E_k \cup E_0) \neq \emptyset.$$

Choose $m \in \mathcal{N} \setminus (E_k \cup E_0)$. Then we have

$$\begin{aligned} \Upsilon(\xi_k; j_k + \varepsilon + \mathcal{H}) &= \Upsilon((\xi_k - y'_m) + y'_m; j_k + \varepsilon + \mathcal{H}) \\ &\geq \min\{\Upsilon(y'_m - \xi_k; j_k + \varepsilon), \Upsilon(y'_m; \mathcal{H})\} \\ &> 1 - \lambda^*, \end{aligned}$$

and similarly,

$$\begin{aligned} \Omega(\xi_k; j_k + \varepsilon + \mathcal{H}) &\leq \max\{\Omega(y'_m - \xi_k; j_k + \varepsilon), \Omega(y'_m; \mathcal{H})\} \\ &< \lambda^*, \\ \Gamma(\xi_k; j_k + \varepsilon + \mathcal{H}) &\leq \max\{\Gamma(y'_m - \xi_k; j_k + \varepsilon), \Gamma(y'_m; \mathcal{H})\} \\ &< \lambda^*. \end{aligned}$$

Hence, for every $k \in \mathbb{N}$,

$$\xi_k \in B(0, j_k + \varepsilon + \mathcal{H}).$$

Therefore, $\{\xi_k\}$ is bounded in $(\mathcal{Y}, \Upsilon, \Omega, \Gamma)$.

By completeness, there exists a subsequence $\{\xi_{k_\ell}\}$ such that

$$\xi_{k_\ell} \rightarrow \xi \quad \text{in } (\mathcal{Y}, \Upsilon, \Omega, \Gamma).$$

That is, for every $\varepsilon > 0$ and $\lambda^* \in (0, 1)$, there exists ℓ_0 such that

$$\Upsilon(\xi_{k_\ell} - \xi; \varepsilon) > 1 - \lambda^*, \quad \Omega(\xi_{k_\ell} - \xi; \varepsilon) < \lambda^*, \quad \Gamma(\xi_{k_\ell} - \xi; \varepsilon) < \lambda^*$$

for all $\ell \geq \ell_0$.

For a fix $\varepsilon > 0$ and $\lambda^* \in (0, 1)$, we obtain

$$\Upsilon(y'_m - \xi; j_0 + \varepsilon) \geq \limsup_{\ell \rightarrow \infty} \Upsilon(y'_m - \xi_{k_\ell}; j_{k_\ell} + \varepsilon),$$

and similarly,

$$\Omega(y'_m - \xi; j_0 + \varepsilon) \leq \liminf_{\ell \rightarrow \infty} \Omega(y'_m - \xi_{k_\ell}; j_{k_\ell} + \varepsilon),$$

$$\Gamma(y'_m - \xi; j_0 + \varepsilon) \leq \liminf_{\ell \rightarrow \infty} \Gamma(y'_m - \xi_{k_\ell}; j_{k_\ell} + \varepsilon).$$

Therefore,

$$\left\{ n \in \mathcal{N} : \frac{1}{n^{\alpha'}} \left\{ m \leq n : \begin{aligned} &\Upsilon(y'_m - \xi; j_0 + \varepsilon) \leq 1 - \lambda^* \\ &\text{or } \Omega(y'_m - \xi; j_0 + \varepsilon) \geq \lambda^* \\ &\text{or } \Gamma(y'_m - \xi; j_0 + \varepsilon) \geq \lambda^* \end{aligned} \right\} \geq \delta^* \right\} \in \mathcal{I}.$$

Thus,

$$\xi \in \text{I-st-LIM}_{j_0}^{\alpha'}(y),$$

which implies

$$\text{I-st-LIM}_{j_0}^{\alpha'}(y) \neq \emptyset.$$

This completes the proof. $\square$

We now prove that rough I-statistical convergence of order α' is stable under small perturbations.

Theorem 3.2. *Let $\{y'_m\}$ and $\{z'_m\}$ be sequences in a neutrosophic normed space such that*

$$\delta_I^*(\{m : \Upsilon(y'_m - z'_m; \varepsilon) \leq 1 - \lambda^*\}) = 0.$$

Then, for any $j' > 0$,

$$y'_m \xrightarrow{j' - \text{I-st}(\alpha')} \xi \iff z'_m \xrightarrow{j' - \text{I-st}(\alpha')} \xi.$$

Proof. We establish the forward implication only, since the converse follows by symmetry.

Assume that

$$y'_m \xrightarrow{j' - \text{I-st}(\alpha')} \xi.$$

Let $\varepsilon > 0$ and $\lambda^* \in (0, 1)$ be arbitrary. By the definition of j' -I-statistical convergence of order α' , we have

$$A = \left\{ m \in \mathbb{N} : \Upsilon(y'_m - \xi; j' + \varepsilon) \leq 1 - \lambda^* \text{ or } \Omega(y'_m - \xi; j' + \varepsilon) \geq \lambda^* \text{ or } \Gamma(y'_m - \xi; j' + \varepsilon) \geq \lambda^* \right\} \in \mathcal{I}.$$

On the other hand, by the assumption

$$\delta_I^*(\{m : \Upsilon(y'_m - z'_m; \varepsilon) \leq 1 - \lambda^*\}) = 0,$$

and similarly for the indeterminacy and falsity functions, it follows that the set

$$B = \left\{ m \in \mathbb{N} : \Upsilon(y'_m - z'_m; \varepsilon) \leq 1 - \lambda^* \text{ or } \Omega(y'_m - z'_m; \varepsilon) \geq \lambda^* \text{ or } \Gamma(y'_m - z'_m; \varepsilon) \geq \lambda^* \right\} \in \mathbf{I}.$$

Since $\mathbf{I}$ is an ideal, it is closed under finite unions. Therefore,

$$A \cup B \in \mathbf{I}.$$

Now, consider any index $m \notin A \cup B$. Then $m \notin A$ and $m \notin B$, so that

$$\Upsilon(y'_m - \xi; j' + \varepsilon) > 1 - \lambda^*, \quad \Omega(y'_m - \xi; j' + \varepsilon) < \lambda^*, \quad \Gamma(y'_m - \xi; j' + \varepsilon) < \lambda^*,$$

and

$$\Upsilon(y'_m - z'_m; \varepsilon) > 1 - \lambda^*, \quad \Omega(y'_m - z'_m; \varepsilon) < \lambda^*, \quad \Gamma(y'_m - z'_m; \varepsilon) < \lambda^*.$$

Using the neutrosophic norm inequality, we get

$$\Upsilon(x + y, s + t) \geq \min\{\Upsilon(x, s), \Upsilon(y, t)\},$$

with

$$x = y'_m - \xi, \quad y = z'_m - y'_m,$$

and noting that

$$z'_m - \xi = (y'_m - \xi) + (z'_m - y'_m),$$

we obtain

$$\begin{aligned} \Upsilon(z'_m - \xi; j' + 2\varepsilon) &= \Upsilon((y'_m - \xi) + (z'_m - y'_m), (j' + \varepsilon) + \varepsilon) \\ &\geq \min\left\{\Upsilon(y'_m - \xi; j' + \varepsilon), \Upsilon(z'_m - y'_m; \varepsilon)\right\}. \end{aligned}$$

Since

$$\Upsilon(z'_m - y'_m; \varepsilon) = \Upsilon(y'_m - z'_m; \varepsilon),$$

it follows that

$$\Upsilon(z'_m - \xi; j' + 2\varepsilon) > 1 - \lambda^*.$$

Similarly,

$$\Omega(x + y, s + t) \leq \max\{\Omega(x, s), \Omega(y, t)\},$$

we obtain

$$\begin{aligned} \Omega(z'_m - \xi; j' + 2\varepsilon) &\leq \max\left\{\Omega(y'_m - \xi; j' + \varepsilon), \Omega(z'_m - y'_m; \varepsilon)\right\} \\ &< \lambda^*. \end{aligned}$$

Again,

$$\Gamma(x + y, s + t) \leq \max\{\Gamma(x, s), \Gamma(y, t)\},$$

we have

$$\begin{aligned} \Gamma(z'_m - \xi; j' + 2\varepsilon) &\leq \max\left\{\Gamma(y'_m - \xi; j' + \varepsilon), \Gamma(z'_m - y'_m; \varepsilon)\right\} \\ &< \lambda^*. \end{aligned}$$

Hence, for every $m \notin A \cup B$,

$$\Upsilon(z'_m - \xi; j' + 2\varepsilon) > 1 - \lambda^*,$$

$$\Omega(z'_m - \xi; j' + 2\varepsilon) < \lambda^*,$$

and

$$\Gamma(z'_m - \xi; j' + 2\varepsilon) < \lambda^*.$$

Consequently,

$$\begin{aligned} &\left\{ m \in \mathbb{N} : \Upsilon(z'_m - \xi; j' + 2\varepsilon) \leq 1 - \lambda^* \text{ or } \Omega(z'_m - \xi; j' + 2\varepsilon) \geq \lambda^* \text{ or } \Gamma(z'_m - \xi; j' + 2\varepsilon) \geq \lambda^* \right\} \\ &\subseteq A \cup B. \end{aligned}$$

Since $A \cup B \in \mathbf{I}$, we conclude that

$$\left\{ m : \Upsilon(z'_m - \xi; j' + 2\varepsilon) \leq 1 - \lambda^* \text{ or } \Omega(z'_m - \xi; j' + 2\varepsilon) \geq \lambda^* \text{ or } \Gamma(z'_m - \xi; j' + 2\varepsilon) \geq \lambda^* \right\} \in \mathbf{I}.$$

As $\varepsilon > 0$ and $\lambda^* \in (0, 1)$ were arbitrary, the definition of j' -I-statistical convergence of order α' yields

$$z'_m \xrightarrow{j'-\mathbf{I}\text{-st}(\alpha')} \xi.$$

The converse implication is proved in exactly the same manner by interchanging the roles of the sequences $\{y'_m\}$ and $\{z'_m\}$. Therefore,

$$y'_m \xrightarrow{j'-\mathbf{I}\text{-st}(\alpha')} \xi \iff z'_m \xrightarrow{j'-\mathbf{I}\text{-st}(\alpha')} \xi.$$

This completes the proof. $\square$

Next, we show that if the intersection of all rough I-statistical limit sets of order α' is nonempty, then the limit point is unique.

Theorem 3.3. *If $\bigcap_{j'>0} \mathbf{I}\text{-st-LIM}_{j'}^{\alpha'}(y) \neq \emptyset$, then the intersection consists of exactly one point.*

Proof. Assume, to the contrary, that there exist two distinct points $\xi_1, \xi_2 \in \mathcal{Y}$ such that

$$\xi_1 \neq \xi_2 \quad \text{and} \quad \xi_1, \xi_2 \in \bigcap_{j'>0} \mathbf{I}\text{-st-LIM}_{j'}^{\alpha'}(y).$$

Fix $\varepsilon > 0$ and $\lambda^* \in (0, 1)$. Since $\xi_1 \in \mathbf{I}\text{-st-LIM}_{j'}^{\alpha'}(y)$, there exists a set $A_{j'} \in \mathbf{I}$ such that for all $m \notin A_{j'}$,

$$\Upsilon(y'_m - \xi_1; j' + \varepsilon) > 1 - \lambda^*, \quad \Omega(y'_m - \xi_1; j' + \varepsilon) < \lambda^*, \quad \Gamma(y'_m - \xi_1; j' + \varepsilon) < \lambda^*. \quad (1)$$

Similarly, since $\xi_2 \in \mathbf{I}\text{-st-LIM}_{j'}^{\alpha'}(y)$, there exists a set $B_{j'} \in \mathbf{I}$ such that for all $m \notin B_{j'}$,

$$\Upsilon(y'_m - \xi_2; j' + \varepsilon) > 1 - \lambda^*, \quad \Omega(y'_m - \xi_2; j' + \varepsilon) < \lambda^*, \quad \Gamma(y'_m - \xi_2; j' + \varepsilon) < \lambda^*. \quad (2)$$

Since $\mathbf{I}$ is an admissible ideal, $A_{j'} \cup B_{j'} \in \mathbf{I}$. Choose $m \notin A_{j'} \cup B_{j'}$. Then combining (1) and (2), we obtain

$$\Upsilon(\xi_1 - \xi_2; 2j' + 2\varepsilon) \geq \min\{\Upsilon(y'_m - \xi_1; j' + \varepsilon), \Upsilon(y'_m - \xi_2; j' + \varepsilon)\} > 1 - \lambda^*,$$

$$\Omega(\xi_1 - \xi_2; 2j' + 2\varepsilon) \leq \max\{\Omega(y'_m - \xi_1; j' + \varepsilon), \Omega(y'_m - \xi_2; j' + \varepsilon)\} < \lambda^*,$$

$$\Gamma(\xi_1 - \xi_2; 2j' + 2\varepsilon) \leq \max\{\Gamma(y'_m - \xi_1; j' + \varepsilon), \Gamma(y'_m - \xi_2; j' + \varepsilon)\} < \lambda^*.$$

Since $j' > 0$ is arbitrary, letting $j' \rightarrow 0^+$, we obtain

$$\Upsilon(\xi_1 - \xi_2; 2\varepsilon) = 1, \quad \Omega(\xi_1 - \xi_2; 2\varepsilon) = 0, \quad \Gamma(\xi_1 - \xi_2; 2\varepsilon) = 0.$$

this implies that

$$\xi_1 - \xi_2 = 0,$$

that is, $\xi_1 = \xi_2$, which contradicts the assumption $\xi_1 \neq \xi_2$.

Hence, the intersection $\bigcap_{j'>0} \mathbf{I}\text{-st-LIM}_{j'}^{\alpha'}(y)$ contains exactly one point. $\square$

We next characterize rough I-statistical convergence of order α' via subsequences indexed by filter sets.

Theorem 3.4. *A sequence $\{y'_m\}$ in $(\mathcal{Y}, \Upsilon, \Omega, \Gamma)$ is rough I-statistically convergent of order α' if and only if it admits a subsequence indexed by a set $M \in \mathcal{F}(\mathbf{I})$ which is rough statistically convergent of order α' .*

Proof. Assume that the sequence $\{y'_m\}$ is rough I-statistically convergent of order α' to ξ with roughness $j' > 0$. Then, by definition, for every $\varepsilon > 0$ and $\lambda^* \in (0, 1)$,

$$\delta_I^* \left(\left\{ m \in \mathcal{N} : \Upsilon(y'_m - \xi; j' + \varepsilon) \leq 1 - \lambda^* \right. \right. \\ \left. \left. \text{or } \Omega(y'_m - \xi; j' + \varepsilon) \geq \lambda^* \right. \right. \\ \left. \left. \text{or } \Gamma(y'_m - \xi; j' + \varepsilon) \geq \lambda^* \right\} \right) = 0. \quad (3)$$

Define the set

$$M = \{m \in \mathcal{N} : \Upsilon(y'_m - \xi; j' + \varepsilon) > 1 - \lambda^*, \\ \Omega(y'_m - \xi; j' + \varepsilon) < \lambda^*, \\ \Gamma(y'_m - \xi; j' + \varepsilon) < \lambda^*\}.$$

From (3), we have

$$\mathcal{N} \setminus M \in \mathbf{I},$$

and hence $M \in \mathcal{F}(\mathbf{I})$.

Now consider the subsequence $\{y'_m\}_{m \in M}$. For each $n \in \mathcal{N}$, define

$$M_n = \{m \leq n : m \in M\}.$$

Then, for all $m \in M_n$, the inequalities

$$\Upsilon(y'_m - \xi; j' + \varepsilon) > 1 - \lambda^*, \quad \Omega(y'_m - \xi; j' + \varepsilon) < \lambda^*, \quad \Gamma(y'_m - \xi; j' + \varepsilon) < \lambda^*$$

hold.

Hence,

$$\frac{1}{n^{\alpha'}} \left| \{m \leq n : m \in M \text{ and } \Upsilon(y'_m - \xi; j' + \varepsilon) \leq 1 - \lambda^* \right. \\ \left. \text{or } \Omega(y'_m - \xi; j' + \varepsilon) \geq \lambda^* \right. \\ \left. \text{or } \Gamma(y'_m - \xi; j' + \varepsilon) \geq \lambda^* \} \right| = 0.$$

Therefore,

$$\lim_{n \rightarrow \infty} \frac{1}{n^{\alpha'}} \left| \{m \leq n : m \in M \text{ and } y'_m \notin B(\xi, j' + \varepsilon, \lambda^*)\} \right| = 0,$$

which shows that the subsequence $\{y'_m\}_{m \in M}$ is rough statistically convergent of order α' to ξ .

Conversely, suppose that there exists a set $M \in \mathcal{F}(\mathbf{I})$ such that the subsequence $\{y'_m\}_{m \in M}$ is rough statistically convergent of order α' to ξ with roughness $j' > 0$.

Then, for every $\varepsilon > 0$ and $\lambda^* \in (0, 1)$,

$$\lim_{n \rightarrow \infty} \frac{1}{n^{\alpha'}} \left| \{m \leq n : m \in M \text{ and } \Upsilon(y'_m - \xi; j' + \varepsilon) \leq 1 - \lambda^* \right. \\ \left. \text{or } \Omega(y'_m - \xi; j' + \varepsilon) \geq \lambda^* \right. \\ \left. \text{or } \Gamma(y'_m - \xi; j' + \varepsilon) \geq \lambda^* \} \right| = 0.$$

Since $M \in \mathcal{F}(\mathbf{I})$, we have $\mathcal{N} \setminus M \in \mathbf{I}$. Let

$$A = \{m \in \mathcal{N} : \Upsilon(y'_m - \xi; j' + \varepsilon) \leq 1 - \lambda^* \text{ or } \Omega(y'_m - \xi; j' + \varepsilon) \geq \lambda^* \text{ or } \Gamma(y'_m - \xi; j' + \varepsilon) \geq \lambda^*\}.$$

Then

$$A \subseteq (\mathcal{N} \setminus M) \cup (A \cap M).$$

By subadditivity of δ_I^* ,

$$\delta_I^*(A) \leq \delta_I^*(\mathcal{N} \setminus M) + \delta_I^*(A \cap M).$$

Since $\mathcal{N} \setminus M \in \mathbf{I}$, $\delta_I^*(\mathcal{N} \setminus M) = 0$, and $\delta_I^*(A \cap M) = 0$. Hence,

$$\delta_I^*(A) = 0.$$

Thus,

$$\delta_I^*\left(\{m \in \mathcal{N} : \Upsilon(y'_m - \xi; j' + \varepsilon) \leq 1 - \lambda^* \text{ or } \Omega(y'_m - \xi; j' + \varepsilon) \geq \lambda^* \text{ or } \Gamma(y'_m - \xi; j' + \varepsilon) \geq \lambda^*\}\right) = 0.$$

This proves that $\{y'_m\}$ is rough I-statistically convergent of order α' to ξ . $\square$

Finally, we present a decomposition characterization of rough I-statistical convergence of order α' .

Theorem 3.5. *A sequence $\{y'_m\}$ is rough I-statistically convergent of order α' if and only if it can be written as $y'_m = u_m + v_m$, where $\{u_m\}$ is I-statistically convergent of order α' and $\{v_m\}$ is I-statistically bounded by j' .*

Proof. Assume that $\{y'_m\}$ is rough I-statistically convergent of order α' to ξ with roughness $j' > 0$. Define sequences $\{u_m\}$ and $\{v_m\}$ by

$$u_m = \xi, \quad v_m = y'_m - \xi \quad (m \in \mathcal{N}).$$

We first show that $\{u_m\}$ is I-statistically convergent of order α' to ξ . Since $u_m - \xi = 0$ for all m , we have for every $\varepsilon > 0$ and $\lambda^* \in (0, 1)$,

$$\Upsilon(u_m - \xi; \varepsilon) = 1, \quad \Omega(u_m - \xi; \varepsilon) = 0, \quad \Gamma(u_m - \xi; \varepsilon) = 0,$$

which immediately implies

$$\delta_I^{\alpha'}\left(\{m : \Upsilon(u_m - \xi; \varepsilon) \leq 1 - \lambda^* \text{ or } \Omega(u_m - \xi; \varepsilon) \geq \lambda^* \text{ or } \Gamma(u_m - \xi; \varepsilon) \geq \lambda^*\}\right) = 0.$$

Hence, $\{u_m\}$ is I-statistically convergent of order α' to ξ .

Next, we show that $\{v_m\}$ is I-statistically bounded of order α' by j' . Since y'_m is rough I-statistically convergent to ξ , for every $\varepsilon > 0$ and $\lambda^* \in (0, 1)$,

$$\delta_I^{\alpha'}\left(\{m : \Upsilon(v_m; j' + \varepsilon) \leq 1 - \lambda^* \text{ or } \Omega(v_m; j' + \varepsilon) \geq \lambda^* \text{ or } \Gamma(v_m; j' + \varepsilon) \geq \lambda^*\}\right) = 0.$$

This proves that $\{v_m\}$ is I-statistically bounded of order α' with bound j' .

Conversely, assume that $y'_m = u_m + v_m$, where $\{u_m\}$ is I-statistically convergent of order α' to ξ and $\{v_m\}$ is I-statistically bounded of order α' with bound j' .

Fix $\varepsilon > 0$ and $\lambda^* \in (0, 1)$. Define

$$A = \{m : \Upsilon(u_m - \xi; \varepsilon) \leq 1 - \lambda^* \text{ or } \Omega(u_m - \xi; \varepsilon) \geq \lambda^* \text{ or } \Gamma(u_m - \xi; \varepsilon) \geq \lambda^*\},$$

$$B = \{m : \Upsilon(v_m; j') \leq 1 - \lambda^* \text{ or } \Omega(v_m; j') \geq \lambda^* \text{ or } \Gamma(v_m; j') \geq \lambda^*\}.$$

Then $A, B \in \mathbf{I}$. Since $\mathbf{I}$ is admissible, $A \cup B \in \mathbf{I}$.

Let $m \notin A \cup B$. Then we obtain

$$\Upsilon(y'_m - \xi; j' + \varepsilon) \geq \min\{\Upsilon(u_m - \xi; \varepsilon), \Upsilon(v_m; j')\} > 1 - \lambda^*,$$

$$\Omega(y'_m - \xi; j' + \varepsilon) \leq \max\{\Omega(u_m - \xi; \varepsilon), \Omega(v_m; j')\} < \lambda^*,$$

$$\Gamma(y'_m - \xi; j' + \varepsilon) \leq \max\{\Gamma(u_m - \xi; \varepsilon), \Gamma(v_m; j')\} < \lambda^*.$$

Therefore,

$$\delta_I^{\alpha'}\left(\{m : \Upsilon(y'_m - \xi; j' + \varepsilon) \leq 1 - \lambda^* \text{ or } \Omega(y'_m - \xi; j' + \varepsilon) \geq \lambda^* \text{ or } \Gamma(y'_m - \xi; j' + \varepsilon) \geq \lambda^*\}\right) = 0,$$

which shows that $\{y'_m\}$ is rough I-statistically convergent of order α' to ξ with roughness j' . $\square$

3.1. Application to Fixed Point Iteration in Neutrosophic Normed Spaces. Let $(\mathcal{Y}, \Upsilon, \Omega, \Gamma)$ be a neutrosophic normed space and let $T : \mathcal{Y} \rightarrow \mathcal{Y}$ be a self-mapping having a fixed point ξ , that is, $T(\xi) = \xi$. Consider the Picard iterative process $y'_{m+1} = T(y'_m)$, $m \in \mathbb{N}$. In practical computations, measurements and rounding errors are unavoidable. Hence, the generated sequence may take the form $z'_m = y'_m + e_m$, where e_m denotes the error sequence.

Suppose that $y'_m \xrightarrow{\text{I-st}(\alpha')} \xi$
and that the error sequence is I-statistically bounded of order α' by some roughness degree $j' > 0$.

Then, by Theorem 3.5, the perturbed iteration sequence

$$z'_m = y'_m + e_m$$

is rough I-statistically convergent of order α' to the same fixed point ξ .

Consequently, even in the presence of neutrosophic uncertainty and statistical perturbations, the iterative algorithm preserves convergence up to the roughness level j' . Therefore, rough I-statistical convergence provides a quantitative measure of stability for fixed-point algorithms in neutrosophic environments.

This application is particularly useful in numerical methods, optimization, artificial intelligence, and decision-making models where exact convergence cannot be guaranteed because of incomplete, indeterminate, or inconsistent information.

Corollary 3.1. *Let $(\mathcal{Y}, \Upsilon, \Omega, \Gamma)$ be a neutrosophic normed space and let $y' * m$ be generated by the Picard iteration*

$$y' * m + 1 = T(y'_m),$$

where T possesses a fixed point ξ .

If y'_m is I-statistically convergent of order α' to ξ and the perturbation sequence e_m is I-statistically bounded of order α' by j' , then the perturbed sequence

$$z'_m = y'_m + e_m$$

is rough I-statistically convergent of order α' to ξ with roughness degree j' .

Proof. Since $\{y'_m\}$ is I-statistically convergent of order α' to the fixed point ξ , it follows that for every $\varepsilon > 0$ and $\lambda^* \in (0, 1)$,

$$\begin{aligned} \delta_I^* \left(\left\{ m \in \mathbb{N} : \Upsilon(y'_m - \xi; \varepsilon) \leq 1 - \lambda^* \right. \right. \\ \text{or } \Omega(y'_m - \xi; \varepsilon) \geq \lambda^* \\ \left. \left. \text{or } \Gamma(y'_m - \xi; \varepsilon) \geq \lambda^* \right\} \right) = 0. \end{aligned} \quad (1)$$

Furthermore, since $\{e_m\}$ is I-statistically bounded of order α' by j' , it follows that for every $\lambda^* \in (0, 1)$,

$$\begin{aligned} \delta_I^* \left(\left\{ m \in \mathbb{N} : \Upsilon(e_m; j') \leq 1 - \lambda^* \right. \right. \\ \text{or } \Omega(e_m; j') \geq \lambda^* \\ \left. \left. \text{or } \Gamma(e_m; j') \geq \lambda^* \right\} \right) = 0. \end{aligned} \quad (2)$$

For fixed $\varepsilon > 0$ and $\lambda^* \in (0, 1)$, we define

$$\begin{aligned} A = \left\{ m \in \mathbb{N} : \Upsilon(y'_m - \xi; \varepsilon) \leq 1 - \lambda^* \right. \\ \text{or } \Omega(y'_m - \xi; \varepsilon) \geq \lambda^* \\ \left. \text{or } \Gamma(y'_m - \xi; \varepsilon) \geq \lambda^* \right\}, \end{aligned}$$

and

$$B = \left\{ m \in \mathbb{N} : \Upsilon(e_m; j') \leq 1 - \lambda^* \right. \\ \left. \text{or } \Omega(e_m; j') \geq \lambda^* \right. \\ \left. \text{or } \Gamma(e_m; j') \geq \lambda^* \right\}.$$

By (1) and (2), we have

$$\delta_I^*(A) = 0, \quad \delta_I^*(B) = 0.$$

Since δ_I^* is subadditive, we obtain

$$\delta_I^*(A \cup B) = 0.$$

Now, let

$$m \notin A \cup B.$$

Then,

$$\Upsilon(y'_m - \xi; \varepsilon) > 1 - \lambda^*, \quad \Omega(y'_m - \xi; \varepsilon) < \lambda^*, \quad \Gamma(y'_m - \xi; \varepsilon) < \lambda^*,$$

and

$$\Upsilon(e_m; j') > 1 - \lambda^*, \quad \Omega(e_m; j') < \lambda^*, \quad \Gamma(e_m; j') < \lambda^*.$$

Since

$$z'_m - \xi = (y'_m - \xi) + e_m,$$

the neutrosophic triangle inequality yields

$$\begin{aligned} \Upsilon(z'_m - \xi; j' + \varepsilon) &= \Upsilon((y'_m - \xi) + e_m, \varepsilon + j') \\ &\geq \min \left\{ \Upsilon(y'_m - \xi; \varepsilon), \Upsilon(e_m; j') \right\} \\ &> 1 - \lambda^*. \end{aligned}$$

Similarly,

$$\begin{aligned} \Omega(z'_m - \xi; j' + \varepsilon) &\leq \max \left\{ \Omega(y'_m - \xi; \varepsilon), \Omega(e_m; j') \right\} \\ &< \lambda^*, \end{aligned}$$

and

$$\begin{aligned} \Gamma(z'_m - \xi; j' + \varepsilon) &\leq \max \left\{ \Gamma(y'_m - \xi; \varepsilon), \Gamma(e_m; j') \right\} \\ &< \lambda^*. \end{aligned}$$

Consequently,

$$C = \left\{ m \in \mathbb{N} : \Upsilon(z'_m - \xi; j' + \varepsilon) \leq 1 - \lambda^* \right. \\ \left. \text{or } \Omega(z'_m - \xi; j' + \varepsilon) \geq \lambda^* \right. \\ \left. \text{or } \Gamma(z'_m - \xi; j' + \varepsilon) \geq \lambda^* \right\} \subseteq A \cup B.$$

Therefore,

$$\delta_I^*(C) \leq \delta_I^*(A \cup B) = 0.$$

Hence,

$$\begin{aligned} \delta_I^* \left(\left\{ m \in \mathbb{N} : \Upsilon(z'_m - \xi; j' + \varepsilon) \leq 1 - \lambda^* \right. \right. \\ \left. \text{or } \Omega(z'_m - \xi; j' + \varepsilon) \geq \lambda^* \right. \\ \left. \left. \text{or } \Gamma(z'_m - \xi; j' + \varepsilon) \geq \lambda^* \right\} \right) = 0. \end{aligned}$$

Since $\varepsilon > 0$ and $\lambda^* \in (0, 1)$ were arbitrary, by the definition of rough I-statistical convergence of order α' , we conclude that

$$z'_m \xrightarrow{j'-I-st(\alpha')} \xi.$$

Therefore, the perturbed Picard sequence $\{z'_m\}$ is rough I-statistically convergent of order α' to the fixed point ξ with roughness degree j' . $\square$

Example 3.3. Consider the neutrosophic normed space $(\mathbb{R}, \Upsilon, \Omega, \Gamma)$, where

$$\Upsilon(x; t) = \frac{t}{t + |x|}, \quad \Omega(x; t) = \frac{|x|}{2(t + |x|)}, \quad \Gamma(x; t) = \frac{|x|}{2(t + |x|)}$$

for every $x \in \mathbb{R}$ and $t > 0$.

Let $\mathbf{I} = \mathbf{I}_f$ be the ideal of all finite subsets of $\mathbb{N}$ and let

$$T(x) = \frac{x}{2}.$$

Clearly, T has the unique fixed point

$$\xi = 0.$$

Starting from $y'_1 = 1$, consider the Picard iteration

$$y'_{m+1} = T(y'_m) = \frac{y'_m}{2}.$$

Then,

$$y'_m = \frac{1}{2^{m-1}}.$$

Since

$$\lim_{m \rightarrow \infty} y'_m = 0,$$

it follows that

$$y'_m \xrightarrow{\mathbf{I}_f-st(1)} 0.$$

Now, we define a perturbation sequence

$$e_m = \begin{cases} 1, & \text{if } m = k^2 \text{ for some } k \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

Let

$$j' = 1.$$

The set of indices for which $e_m \neq 0$ is

$$A = \{1, 4, 9, 16, \dots\}.$$

Since

$$|A \cap \{1, \dots, n\}| = \lfloor \sqrt{n} \rfloor,$$

we obtain

$$\lim_{n \rightarrow \infty} \frac{\lfloor \sqrt{n} \rfloor}{n} = 0.$$

Hence, $\{e_m\}$ is $\mathbf{I}_f$ -statistically bounded of order $\alpha' = 1$ by $j' = 1$.

Define the perturbed sequence

$$z'_m = y'_m + e_m.$$

Then,

$$z'_m = \begin{cases} \frac{1}{2^{m-1}} + 1, & \text{if } m = k^2, \\ \frac{1}{2^{m-1}}, & \text{otherwise.} \end{cases}$$

Observe that

$$z'_{k^2} = 1 + \frac{1}{2^{k^2-1}} \longrightarrow 1,$$

whereas

$$z'_m = \frac{1}{2^{m-1}} \longrightarrow 0$$

for non-square indices. Consequently, the sequence $\{z'_m\}$ is not ordinarily convergent.

However, the exceptional terms occur only on the set of square numbers, which has natural density zero. Therefore, by the above corollary,

$$z'_m \xrightarrow{1-\mathbf{I}_f\text{-st}(1)} 0.$$

Hence,

$$0 \in \mathbf{I}_f\text{-st-LIM}_1^1(z).$$

Thus, although the perturbations destroy ordinary convergence, the perturbed Picard iteration remains rough $\mathbf{I}$ -statistically convergent to the fixed point $\xi = 0$ with roughness degree 1.

3.2. Application to Stability Analysis of Neutrosophic Dynamical Systems. Consider a discrete-time neutrosophic dynamical system

$$x_{m+1} = T(x_m) + e_m,$$

where $T : \mathcal{Y} \rightarrow \mathcal{Y}$ is a system operator and $\{e_m\}$ denotes external disturbances.

In practical situations, disturbances may contain rare but arbitrarily large impulses caused by transmission failures, abrupt environmental changes, or incomplete information. Such impulses generally prevent ordinary convergence of trajectories.

Suppose that the disturbance sequence satisfies

$$\delta_I^*\left(\{m : \|e_m\| > \varepsilon\}\right) = 0$$

for every $\varepsilon > 0$. Then, the trajectory sequence $\{x_m\}$ may not converge classically, but can still possess a rough $\mathbf{I}$ -statistical limit of order α' . If

$$x_m \xrightarrow{j'-\mathbf{I}\text{-st}(\alpha')} x^*,$$

then x^* represents an asymptotically stable neutrosophic equilibrium up to a tolerance level j' .

The parameter j' quantifies the maximum admissible uncertainty radius, while the order parameter α' controls the rate at which exceptional perturbations become negligible.

Therefore, rough $\mathbf{I}$ -statistical convergence of order α' provides a new tool for studying practical stability, robustness, and long-term behavior of uncertain dynamical systems modeled in neutrosophic normed spaces.

Theorem 3.6. *Let $(\mathcal{Y}, \Upsilon, \Omega, \Gamma)$ be a neutrosophic normed space and consider the discrete-time dynamical system $x_{m+1} = T(x_m) + e_m$, where $T : \mathcal{Y} \rightarrow \mathcal{Y}$ is an operator and e_m is a disturbance sequence. Assume that:*

- (i) $x^* \in \mathcal{Y}$ is a fixed point of T , that is, $T(x^*) = x^*$;
- (ii) the disturbance sequence is $\mathbf{I}$ -statistically bounded of order α' by $j' > 0$;
- (iii) the unperturbed trajectory $y_{m+1} = T(y_m)$ is $\mathbf{I}$ -statistically convergent of order α' to x^* .

Then the perturbed trajectory x_m is rough $\mathbf{I}$ -statistically convergent of order α' to x^* with roughness degree j' , that is,

$$x_m \xrightarrow{j'-\mathbf{I}\text{-st}(\alpha')} x^*.$$

Consequently, x^* is a rough $\mathbf{I}$ -statistically stable neutrosophic equilibrium of the system.

Proof. Let

$$v_m = x_m - y_m,$$

so that

$$x_m = y_m + v_m \quad (m \in \mathbb{N}).$$

By assumption (iii), for every $\varepsilon > 0$ and $\lambda^* \in (0, 1)$,

$$\delta_I^* \left(\{m \in \mathbb{N} : \Upsilon(y_m - x^*; \varepsilon) \leq 1 - \lambda^* \text{ or } \Omega(y_m - x^*; \varepsilon) \geq \lambda^* \text{ or } \Gamma(y_m - x^*; \varepsilon) \geq \lambda^*\} \right) = 0.$$

Since $\{e_m\}$ is I-statistically bounded of order α' by j' , it follows that

$$\delta_I^* \left(\{m \in \mathbb{N} : \Upsilon(v_m; j') \leq 1 - \lambda^* \text{ or } \Omega(v_m; j') \geq \lambda^* \text{ or } \Gamma(v_m; j') \geq \lambda^*\} \right) = 0.$$

We define

$$A = \left\{ m \in \mathbb{N} : \Upsilon(y_m - x^*; \varepsilon) \leq 1 - \lambda^* \text{ or } \Omega(y_m - x^*; \varepsilon) \geq \lambda^* \text{ or } \Gamma(y_m - x^*; \varepsilon) \geq \lambda^* \right\},$$

and

$$B = \left\{ m \in \mathbb{N} : \Upsilon(v_m; j') \leq 1 - \lambda^* \text{ or } \Omega(v_m; j') \geq \lambda^* \text{ or } \Gamma(v_m; j') \geq \lambda^* \right\}.$$

Then,

$$\delta_I^*(A) = 0, \quad \delta_I^*(B) = 0,$$

and therefore

$$\delta_I^*(A \cup B) = 0.$$

Let $m \notin A \cup B$. Then,

$$\Upsilon(y_m - x^*; \varepsilon) > 1 - \lambda^*, \quad \Omega(y_m - x^*; \varepsilon) < \lambda^*, \quad \Gamma(y_m - x^*; \varepsilon) < \lambda^*,$$

and

$$\Upsilon(v_m; j') > 1 - \lambda^*, \quad \Omega(v_m; j') < \lambda^*, \quad \Gamma(v_m; j') < \lambda^*.$$

Since

$$x_m - x^* = (y_m - x^*) + v_m,$$

the axioms of a neutrosophic norm imply

$$\begin{aligned} \Upsilon(x_m - x^*; j' + \varepsilon) &\geq \min \left\{ \Upsilon(y_m - x^*; \varepsilon), \Upsilon(v_m; j') \right\} \\ &> 1 - \lambda^*, \end{aligned}$$

$$\begin{aligned} \Omega(x_m - x^*; j' + \varepsilon) &\leq \max \left\{ \Omega(y_m - x^*; \varepsilon), \Omega(v_m; j') \right\} \\ &< \lambda^*, \end{aligned}$$

and

$$\begin{aligned} \Gamma(x_m - x^*; j' + \varepsilon) &\leq \max \left\{ \Gamma(y_m - x^*; \varepsilon), \Gamma(v_m; j') \right\} \\ &< \lambda^*. \end{aligned}$$

Hence,

$$\left\{ m \in \mathbb{N} : \Upsilon(x_m - x^*; j' + \varepsilon) \leq 1 - \lambda^* \text{ or } \Omega(x_m - x^*; j' + \varepsilon) \geq \lambda^* \text{ or } \Gamma(x_m - x^*; j' + \varepsilon) \geq \lambda^* \right\} \subseteq A \cup B.$$

Therefore,

$$\begin{aligned} \delta_I^* \left(\left\{ m \in \mathbb{N} : \Upsilon(x_m - x^*; j' + \varepsilon) \leq 1 - \lambda^* \right. \right. \\ \left. \left. \text{or } \Omega(x_m - x^*; j' + \varepsilon) \geq \lambda^* \right. \right. \\ \left. \left. \text{or } \Gamma(x_m - x^*; j' + \varepsilon) \geq \lambda^* \right\} \right) = 0. \end{aligned}$$

Since $\varepsilon > 0$ and $\lambda^* \in (0, 1)$ are arbitrary, we obtain

$$x_m \xrightarrow{j' - \text{I-st}(\alpha')} x^*.$$

Thus, x^* is a rough I-statistically stable equilibrium point of the neutrosophic dynamical system. $\square$

3.3. Application to Neutrosophic Sensor Networks and Data Fusion. Modern intelligent systems frequently collect information from large-scale sensor networks operating in uncertain environments. Examples include environmental monitoring, smart cities, medical diagnosis, industrial automation, and autonomous vehicles. In such systems, sensor measurements are often contaminated by random noise, communication failures, and occasional abnormal observations.

Classical convergence methods are inadequate for analyzing such data because a small number of extreme measurements may destroy ordinary convergence. Furthermore, uncertainty in sensor readings is naturally represented through the three components of a neutrosophic norm: truth-membership, indeterminacy, and falsity.

Let y'_m denote the fused output of a sensor network at time m and suppose that ξ represents the actual physical quantity being monitored. Assume that the measurements satisfy

$$y'_m = \xi + \eta_m + \zeta_m,$$

where

- (1) η_m is a small stochastic perturbation,
- (2) ζ_m represents rare abnormal measurements caused by sensor malfunction, communication loss, or external disturbances.

If the set of abnormal observations

$$A = \{m : \zeta_m \neq 0\}$$

has I-density of order α' equal to zero, then the sequence $\{y'_m\}$ may fail to converge in the ordinary neutrosophic sense, but it can still be rough I-statistically convergent of order α' .

Consequently, there exists a roughness parameter $j' > 0$ such that

$$y'_m \xrightarrow{j' - \text{I-st}(\alpha')} \xi.$$

This means that almost all sensor outputs remain within a neutrosophic tolerance radius j' of the true state ξ , while the exceptional measurements belong only to an I-small set.

Therefore, the rough I-statistical limit set

$$\text{I-st-LIM}_{j'}^{\alpha'}(y)$$

provides a reliable estimate of the actual system state even in the presence of indeterminate and sporadic faulty observations.

Hence, rough I-statistical convergence of order α' offers a mathematically rigorous framework for robust data fusion, fault-tolerant sensor analysis, and uncertainty-aware decision making in neutrosophic environments.

Theorem 3.7. *Let $(\mathcal{Y}, \Upsilon, \Omega, \Gamma)$ be a neutrosophic normed space and let $\{y'_m\}$ be a sequence of fused sensor measurements given by*

$$y'_m = \xi + \eta_m + \zeta_m,$$

where $\xi \in \mathcal{Y}$ is the true state of the monitored system, $\{\eta_m\}$ is a measurement-noise sequence, and $\{\zeta_m\}$ is an abnormal-error sequence.

Assume that

- (i) $\eta_m \xrightarrow{\text{I-st}(\alpha')} 0$;
- (ii) *the set $A = \{m \in \mathbb{N} : \zeta_m \neq 0\}$ has I-density of order α' equal to zero;*

(iii) there exists $j' > 0$ such that $\{\zeta_m\}$ is I-statistically bounded of order α' by j' .

Then, the fused output sequence $\{y'_m\}$ is rough I-statistically convergent of order α' to ξ , namely

$$y'_m \xrightarrow{j'-\text{I-st}(\alpha')} \xi.$$

Consequently,

$$\xi \in \text{I-st-LIM}_{j'}^{\alpha'}(y),$$

and the sensor network provides a robust estimate of the true state despite sporadic abnormal observations.

Proof. Define

$$u_m = \xi + \eta_m, \quad v_m = \zeta_m.$$

Then,

$$y'_m = u_m + v_m.$$

Since

$$\eta_m \xrightarrow{\text{I-st}(\alpha')} 0,$$

it follows immediately that

$$u_m = \xi + \eta_m \xrightarrow{\text{I-st}(\alpha')} \xi.$$

Indeed, for every $\varepsilon > 0$ and $\lambda^* \in (0, 1)$,

$$\delta_I^*\left(\{m : \Upsilon(\eta_m; \varepsilon) \leq 1 - \lambda^* \text{ or } \Omega(\eta_m; \varepsilon) \geq \lambda^* \text{ or } \Gamma(\eta_m; \varepsilon) \geq \lambda^*\}\right) = 0,$$

which is equivalent to

$$\delta_I^*\left(\{m : \Upsilon(u_m - \xi; \varepsilon) \leq 1 - \lambda^* \text{ or } \Omega(u_m - \xi; \varepsilon) \geq \lambda^* \text{ or } \Gamma(u_m - \xi; \varepsilon) \geq \lambda^*\}\right) = 0.$$

Hence, $\{u_m\}$ is I-statistically convergent of order α' to ξ .

Next, by assumption (iii), the abnormal-error sequence $v_m = \zeta_m$ is I-statistically bounded of order α' by j' . Thus, for every $\lambda^* \in (0, 1)$ there exists $\mathcal{H} > 0$ such that

$$\left\{n \in \mathbb{N} : \frac{1}{n^{\alpha'}} \left| \left\{m \leq n : \Upsilon(v_m; \mathcal{H}) \leq 1 - \lambda^* \text{ or } \Omega(v_m; \mathcal{H}) \geq \lambda^* \text{ or } \Gamma(v_m; \mathcal{H}) \geq \lambda^*\right\} \right| \geq \delta^* \right\} \in \mathbb{I}.$$

Therefore, $\{v_m\}$ is I-statistically bounded of order α' .

Since

$$y'_m = u_m + v_m,$$

where $\{u_m\}$ is I-statistically convergent of order α' to ξ and $\{v_m\}$ is I-statistically bounded of order α' by j' , By the Theorem 3.5 yields

$$y'_m \xrightarrow{j'-\text{I-st}(\alpha')} \xi.$$

Equivalently, for every $\varepsilon > 0$ and $\lambda^* \in (0, 1)$,

$$\delta_I^*\left(\{m : \Upsilon(y'_m - \xi; j' + \varepsilon) \leq 1 - \lambda^* \text{ or } \Omega(y'_m - \xi; j' + \varepsilon) \geq \lambda^* \text{ or } \Gamma(y'_m - \xi; j' + \varepsilon) \geq \lambda^*\}\right) = 0.$$

Hence,

$$\xi \in \text{I-st-LIM}_{j'}^{\alpha'}(y).$$

Therefore, the fused sensor output remains asymptotically inside a neutrosophic tolerance region of radius j' around the true state ξ , except on an I-small set of indices of order α' . This proves the robustness of the data-fusion process against sporadic faulty measurements. $\square$

Example 3.4. Consider the neutrosophic normed space $(\mathbb{R}, \Upsilon, \Omega, \Gamma)$, where

$$\Upsilon(x; t) = \frac{t}{t + |x|}, \quad \Omega(x; t) = \frac{|x|}{2(t + |x|)}, \quad \Gamma(x; t) = \frac{|x|}{2(t + |x|)}$$

for all $x \in \mathbb{R}$ and $t > 0$.

Let $\mathbf{I} = \mathbf{I}_f$ be the ideal of all finite subsets of $\mathbb{N}$ and let the true state of the monitored system be $\xi = 10$.

Define the measurement-noise sequence

$$\eta_m = \frac{(-1)^m}{m},$$

and the abnormal-error sequence

$$\zeta_m = \begin{cases} \sqrt{m}, & \text{if } m = k^2 \text{ for some } k \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

The fused sensor output is

$$y'_m = 10 + \frac{(-1)^m}{m} + \zeta_m.$$

We verify the assumptions of the theorem 3.7.

Step 1. Since $\lim_{m \rightarrow \infty} \frac{(-1)^m}{m} = 0$, it follows that $\eta_m \xrightarrow{\mathbf{I}_f\text{-st}(1)} 0$.

Step 2. The set of abnormal observations is

$$A = \{1, 4, 9, 16, \dots\}.$$

Since

$$|A \cap \{1, \dots, n\}| = \lfloor \sqrt{n} \rfloor,$$

we obtain

$$\lim_{n \rightarrow \infty} \frac{\lfloor \sqrt{n} \rfloor}{n} = 0.$$

Hence, A has $\mathbf{I}_f$ -density of order 1 equal to zero.

Step 3. Choose $j' = 1$. For every non-square index m ,

$$\zeta_m = 0.$$

Therefore,

$$\Upsilon(\zeta_m; 1) = 1, \quad \Omega(\zeta_m; 1) = 0, \quad \Gamma(\zeta_m; 1) = 0.$$

The exceptional indices are precisely the square numbers. Consequently, for every $\lambda^* \in (0, 1)$,

$$\left| \{m \leq n : \Upsilon(\zeta_m; 1) \leq 1 - \lambda^* \text{ or } \Omega(\zeta_m; 1) \geq \lambda^* \text{ or } \Gamma(\zeta_m; 1) \geq \lambda^*\} \right| \leq \sqrt{n}.$$

Hence,

$$\frac{\sqrt{n}}{n} \longrightarrow 0,$$

which proves that $\{\zeta_m\}$ is $\mathbf{I}_f$ -statistically bounded of order 1 by $j' = 1$.

All assumptions of the theorem are satisfied. Therefore,

$$y'_m \xrightarrow{1 - \mathbf{I}_f\text{-st}(1)} 10.$$

Thus,

$$10 \in \mathbf{I}_f\text{-st-LIM}_1^1(y).$$

Notice that the sequence $\{y'_m\}$ is not ordinarily convergent because

$$y'_{k^2} = 10 + \frac{(-1)^{k^2}}{k^2} + k \longrightarrow \infty.$$

Nevertheless, the abnormal measurements occur only on a set of natural density zero. Hence the sensor network still recovers the true state $\xi = 10$ in the sense of rough I-statistical convergence. This illustrates the robustness of neutrosophic data fusion against sporadic large sensor failures.

4. CONCLUSION

We investigated rough I-statistical convergence of order α' in the setting of neutrosophic normed spaces. We established the existence of a minimal roughness parameter for which the corresponding I-statistical limit set is nonempty and showed that this convergence is preserved under small perturbations of sequences. It was further proved that whenever the intersection of all rough I-statistical limit sets is nonempty, it consists of a unique point. Additionally, we characterized rough I-statistical convergence of order α' in terms of subsequences indexed by filter sets and by decomposing sequences into I-statistically convergent and I-statistically bounded parts.

The proposed framework extends classical concepts of rough, statistical, and ideal convergence to the neutrosophic setting. Furthermore, applications to neutrosophic dynamical systems, perturbed Picard iterations, and sensor-network data fusion demonstrate the usefulness of the theory in studying stability, robustness, and uncertainty. Future work may focus on extensions to operator theory, fixed-point theory, sequence spaces, and other neutrosophic structures.

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FUTURE RESEARCH DIRECTIONS IN NEUTROSOPHIC MATHEMATICS

7.1 Introduction

Since its introduction, neutrosophic theory has evolved from a conceptual framework for modeling truth, indeterminacy, and falsity into a broad mathematical discipline encompassing logic, set theory, graph theory, algebra, topology, analysis, decision sciences, artificial intelligence, and engineering. The rapid expansion of neutrosophic mathematics reflects the growing need for mathematical tools capable of representing uncertainty, inconsistency, incompleteness, and ambiguity in increasingly complex systems.

The studies presented in this volume demonstrate the versatility of neutrosophic methods across graph-theoretical, algebraic, and analytical domains. However, the current stage of development represents only the beginning of a much broader scientific journey. Emerging technologies, data-intensive systems, ecological challenges, and quantum information sciences create new opportunities for the application and further development of neutrosophic mathematical structures.

This concluding chapter highlights several promising research directions that may significantly influence the future evolution of neutrosophic mathematics.

7.2 Neutrosophic Artificial Intelligence

Artificial Intelligence (AI) increasingly operates in environments characterized by incomplete information, conflicting evidence, uncertain observations, and rapidly changing conditions. Traditional probabilistic and fuzzy approaches have achieved remarkable success, yet they often struggle when confronted with simultaneously contradictory and indeterminate information.

Neutrosophic Artificial Intelligence offers a promising alternative framework by explicitly incorporating three independent components: Truth degree (T), Indeterminacy degree (I), Falsity degree (F), into reasoning and decision-making processes. Potential research directions include:

Neutrosophic Expert Systems

Development of expert systems capable of handling conflicting knowledge bases while preserving uncertainty information throughout the inference process.

Neutrosophic Knowledge Representation

Construction of knowledge graphs and semantic networks in which concepts and relationships possess neutrosophic attributes.

Neutrosophic Reasoning Engines

Extension of logical inference mechanisms to support indeterminate and contradictory information without forcing premature resolution.

Autonomous Decision-Making

Integration of neutrosophic decision models into autonomous systems such as intelligent robots, autonomous vehicles, industrial automation systems, and adaptive control architectures. The combination of neutrosophic logic and explainable AI may provide more transparent and robust decision-support mechanisms for critical applications.

7.3 Neutrosophic Machine Learning

Machine learning algorithms rely heavily on data quality. In practice, datasets often contain missing observations, contradictory labels, sensor errors, uncertain measurements, and adversarial perturbations. These challenges motivate the development of neutrosophic machine learning frameworks.

Neutrosophic Neural Networks

Future research may investigate neural architectures whose weights, activations, or outputs are represented through neutrosophic components. Such models may better represent uncertainty propagation across deep learning systems.

Neutrosophic Feature Selection

Feature relevance may be expressed through truth, indeterminacy, and falsity measures, leading to more robust dimensionality-reduction techniques.

Neutrosophic Clustering

Many clustering algorithms require precise cluster assignments. Neutrosophic clustering can represent membership, uncertainty, rejection simultaneously, improving performance in noisy environments.

Neutrosophic Reinforcement Learning

Agents operating in uncertain environments may benefit from reward structures incorporating indeterminacy and conflicting information. Potential applications include intelligent transportation systems, smart agriculture, cybersecurity, medical diagnostics, and autonomous exploration.

7.4 Quantum Neutrosophic Structures

One of the most exciting future directions lies at the intersection of neutrosophic mathematics and quantum science. Quantum systems naturally exhibit uncertainty, superposition, probabilistic behavior, contextuality. Many researchers have observed conceptual parallels between quantum indeterminacy and neutrosophic indeterminacy. Potential research topics include:

Quantum Neutrosophic Sets

Development of neutrosophic membership structures defined over quantum state spaces.

Neutrosophic Hilbert Spaces

Extension of classical Hilbert-space theory through neutrosophic-valued inner products and operators.

Neutrosophic Quantum Logic

Construction of logical systems capable of simultaneously representing quantum uncertainty, measurement ambiguity, contradictory observations.

Quantum Information Theory

Applications to quantum communication, quantum cryptography, quantum computing, and quantum decision systems. These developments may eventually contribute to a unified mathematical framework connecting uncertainty theory with quantum phenomena.

7.5 Neutrosophic Network Science

Modern society increasingly depends on large-scale interconnected systems: communication networks, transportation systems, social networks, ecological networks, and energy grids. Such networks often contain uncertain and dynamically changing relationships. The neutrosophic graph structures investigated in this volume provide an important foundation for future advances.

Dynamic Neutrosophic Networks

Modeling networks whose truth, indeterminacy, and falsity values evolve over time.

Multilayer Neutrosophic Graphs

Representing complex systems with multiple interacting layers of relationships.

Resilience and Robustness Analysis

Investigating network vulnerability under uncertain conditions.

Neutrosophic Community Detection

Identifying partially overlapping and uncertain communities within large networks.

Neutrosophic Network Optimization

Designing routing, allocation, and scheduling algorithms capable of operating under uncertainty.

Applications may include: smart cities, energy systems, communication infrastructures, emergency management networks, and ecological monitoring systems.

7.6 Algebraic Topology under Neutrosophic Uncertainty

Topology studies structures preserved under continuous deformation. Extending topological concepts to neutrosophic settings represents a largely unexplored research area.

Potential developments include:

Neutrosophic Topological Spaces

Generalized open and closed sets characterized by neutrosophic membership functions.

Neutrosophic Homology and Cohomology

Development of algebraic invariants capable of measuring uncertainty in geometric structures.

Neutrosophic Simplicial Complexes

Modeling uncertain spatial and relational data.

Persistent Neutrosophic Topology

Combining topological data analysis with neutrosophic uncertainty measures.

These tools may prove valuable in: data science, image processing, biological systems, social network analysis, and geographic information systems.

The integration of algebraic topology with neutrosophic theory may create an entirely new branch of mathematical research.

7.7 Applications in Ecological Systems and Sustainability

Environmental systems are among the most complex and uncertain systems studied today. Climate variability, biodiversity loss, ecosystem interactions, and human impacts generate substantial uncertainty in ecological modeling. Neutrosophic mathematics offers powerful tools for addressing these challenges.

Ecosystem Modeling

Species interactions may be represented using neutrosophic graphs where ecological relationships are uncertain or partially observed.

Biodiversity Assessment

Truth, indeterminacy, and falsity measures can describe the reliability of ecological observations and species distributions.

Climate Risk Analysis

Neutrosophic approaches may support more flexible assessments of future climate scenarios.

Sustainable Resource Management

Applications include forest management, pasture management, watershed conservation, wildlife protection, ecosystem restoration.

Environmental Decision Support Systems

Neutrosophic multi-criteria decision-making methods can assist policymakers in balancing ecological, economic, and social objectives.

7.8 Interdisciplinary Perspectives

Future advances will likely emerge from collaborations across disciplines.

Promising interdisciplinary directions include: Neutrosophic bioinformatics, Neutrosophic medicine, Neutrosophic economics, Neutrosophic cybersecurity, Neutrosophic operations research, Neutrosophic social sciences, Neutrosophic digital twins, Neutrosophic smart-city systems.

The flexibility of neutrosophic representations makes them particularly suitable for problems involving heterogeneous, incomplete, and evolving information.

7.9 Final Remarks

The future of neutrosophic mathematics lies not only in the extension of existing theories but also in the creation of entirely new mathematical structures and interdisciplinary applications. Advances in artificial intelligence, machine learning, quantum sciences, network theory, topology, and sustainability research provide fertile ground for future exploration.

The continued development of neutrosophic theory promises to deepen our understanding of uncertainty and to provide powerful tools for addressing some of the most challenging scientific and societal problems of the twenty-first century.

Mukhtar Ahmad • Florentin Smarandache

Advanced Topics in Neutrosophic Mathematics:

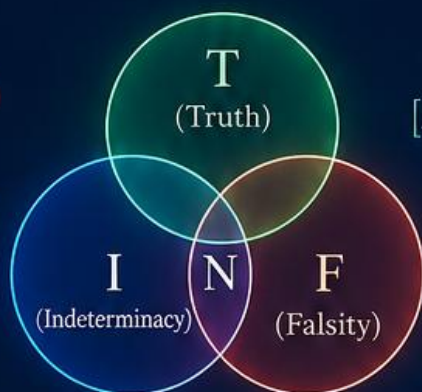
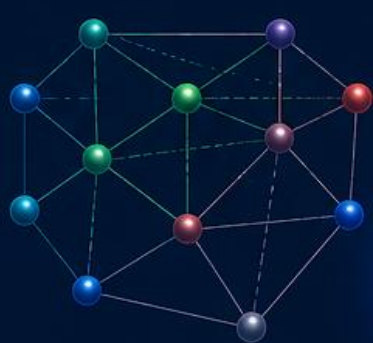
Graphs, Algebra, Analysis, and Applications

This book presents recent advances in neutrosophic mathematics, covering graph theory, algebra, metric spaces, and convergence theory together with their applications in optimization, dynamic systems, and decision modeling.

New models of neutrosophic graphs, including Min-Max and bidirected structures, are introduced and applied to network analysis. Neutrosophic Lie subalgebras and ideals are studied through homomorphisms, images, inverse images, and quotient structures. In analysis, new fixed point results in neutrosophic metric spaces and rough I-statistical convergence of order α' in neutrosophic normed spaces are developed and applied to dynamic programming, iterative processes, and data fusion.

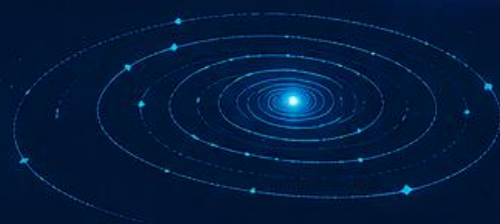
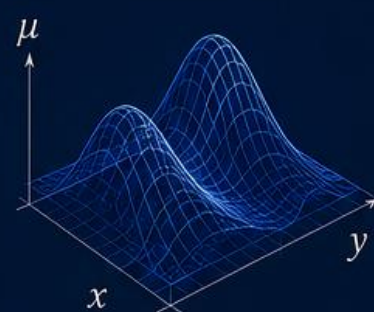
The volume concludes with future research directions such as neutrosophic artificial intelligence, machine learning, quantum neutrosophic structures, network science, algebraic topology under uncertainty, and sustainability applications.

This book is a valuable resource for researchers and graduate students working in uncertainty modeling and advanced mathematical sciences.



$$\begin{aligned} [x, y] &= -[y, x] \\ [x, [y, z]] + [y, [z, x]] \\ &+ [z, [x, y]] = 0 \end{aligned}$$

$$\phi(L) = \begin{pmatrix} T_1 & I_1 & F_1 \\ T_2 & I_2 & F_2 \\ T_3 & I_3 & F_3 \end{pmatrix}$$



$$\begin{aligned} M(T_p(x, y), t) &\geq 1 - \varphi(t, d(x, y)) \\ &\text{for all } t > 0 \end{aligned}$$



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